Oppg. 1

$$4x^2 - 16y^2 = 64$$

$$\frac{x^2}{4^2} - \frac{y^2}{2^2} = 1$$

$\Rightarrow$ Liggende hyperbel:

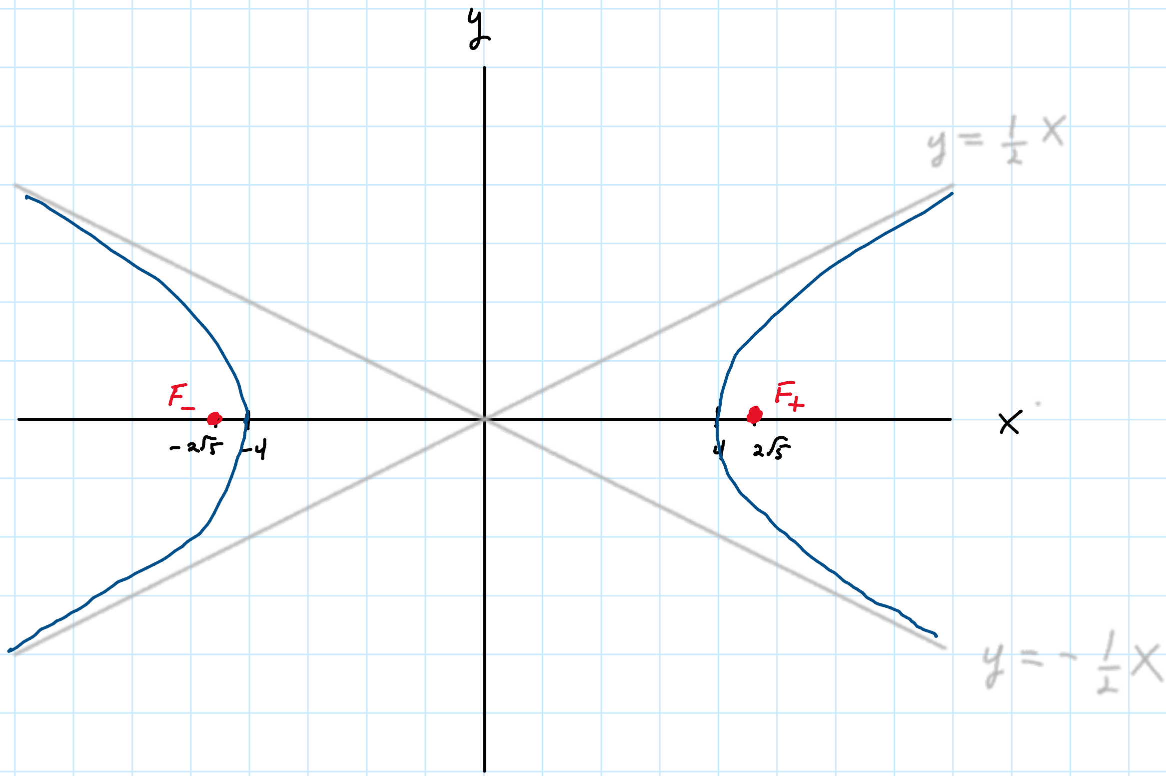
$a = 4$ og $b = 2 \Rightarrow c = \sqrt{a^2 + b^2} = 2\sqrt{5}$

Brennpunkter: $F_{\pm}\,(\pm 2\sqrt{5},\ 0)$

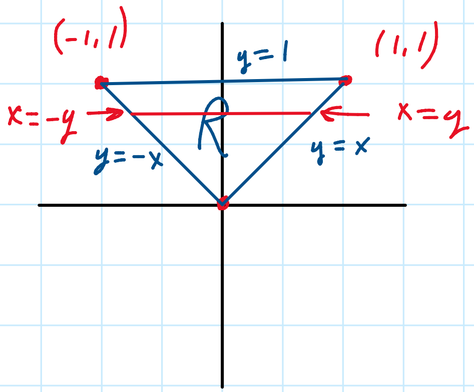
Asymptoter:

$$4x^2 - 16y^2 = 0$$

$$\Rightarrow y = \pm \frac{1}{2} x$$



Oppg. 2



$$\iint_R f(x,y)\,dA = \int_0^1 \int_{-y}^{y} x^2+y^2\,dx\,dy = \int_0^1 \left[\frac{1}{3}x^3 + xy^2\right]_{x=-y}^{x=y} dy$$

$$= \int_0^1 \frac{4}{3}y^3 - \left(-\frac{4}{3}y^3\right) dy = \frac{8}{3}\int_0^1 y^3\,dy$$

$$= \frac{2}{3}\left[y^4\right]_0^1 = \underline{\underline{\frac{2}{3}}}$$

R: $0 \le y \le 1$

$-y \le x \le y$

Merk:

R er en trekant med areal:

$$A_R = \frac{gh}{2} = \frac{2\cdot 1}{2} = 1$$

Siden R har areal 1 så er dette også middelverdien til f:

$$\bar{f} = \frac{1}{A_R}\int_R f(x,y)\,dA = \frac{1}{1}\cdot\frac{2}{3} = \underline{\underline{\frac{2}{3}}}$$

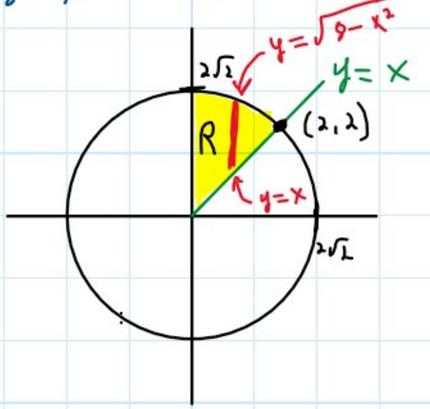
Oppg. 3

$$\int_0^2 \int_x^{\sqrt{8-x^2}} y^2 \, dy \, dx$$

R: $0 \le x \le 2$

$x \le y \le \sqrt{8-x^2}$

Så y er avgrenset nedenfra av linjen $y = x$ og ovenfra av sirkelen $y = \sqrt{8-x^2} \Rightarrow x^2+y^2 = (2\sqrt{2})$



R i polar koor: $\frac{\pi}{4} \le \theta \le \frac{\pi}{2}$

$0 \le r \le 2\sqrt{2}$

$$\Rightarrow \int_0^2 \int_x^{\sqrt{8-x^2}} y^2 \, dy \, dx = \int_{\pi/4}^{\pi/2} \int_0^{2\sqrt{2}} (r \cdot \sin\theta)^2 \, r \, dr \, d\theta$$

$$= \frac{1}{4} \int_{\pi/4}^{\pi/2} \sin^2\theta \left[r^4 \right]_0^{2\sqrt{2}} d\theta = 16 \int_{\pi/4}^{\pi/2} \sin^2\theta \, d\theta$$

$$= 8 \int_{\pi/4}^{\pi/2} 1 - \cos 2\theta \, d\theta = \left[8\theta - 4 \sin 2\theta \right]_{\pi/4}^{\pi/2}$$

$$= \left(8 \cdot \frac{\pi}{2} - 4 \cdot \sin\pi\right) - \left(8 \cdot \frac{\pi}{4} - 4 \cdot \sin\frac{\pi}{2}\right)$$

$$= 4\pi - 0 - 2\pi + 4$$

$$= \underline{\underline{4 + 2\pi}}$$

Oppg. 4a

$$\mathbf{F} = kxyz\,\mathbf{i} + x^2z\,\mathbf{j} + (x^2y + 4z)\,\mathbf{k}$$

a) Divergensen til $\vec{F}$:

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(kxyz) + \frac{\partial}{\partial y}(x^2z) + \frac{\partial}{\partial z}(x^2y + 4z)$$

$$= \underline{\underline{kyz + 4}}$$

Virvlingen til $\vec{F}$:

$$\nabla \times \vec{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial_x & \partial_y & \partial_z \\ kxyz & x^2z & x^2y+4z \end{vmatrix} = \underline{i}\left[\partial_y(x^2y+4z) - \partial_z(x^2z)\right]$$

$$-\underline{j}\left[\partial_x(x^2y+4z) - \partial_z(kxyz)\right]$$

$$+\underline{k}\left[\partial_x(x^2z) - \partial_y(kxyz)\right]$$

$$= \underline{i}\,(x^2 - x^2) - \underline{j}\,(2xy - kxy) + \underline{k}\,(2xz - kxz)$$

$$= \underline{\underline{(k-2)xy\,\underline{j} + (2-k)xz\,\underline{k}}}$$

$k = 2 \Rightarrow \nabla \times \vec{F} = \vec{0}$

Siden $\vec{F}$ er definert på hele $\mathbb{R}^3$, som er sammenhengende og enkelt sammenhengende, kan vi dermed konkludere at $\vec{F}$ er konservativt når $k = 2$.

Oppg. 4b

b) $\nabla f = \vec{F}$ gir likningene:

I) $\frac{\partial f}{\partial x} = 2xyz$

II) $\frac{\partial f}{\partial y} = x^2 z$

III) $\frac{\partial f}{\partial z} = x^2 y + 4z$

Integrerer I:

$f(x,y,z) = x^2 y z + g(y,z)$

Deriverer mhp y:

$$\frac{\partial f}{\partial y} = x^2 z + \frac{\partial}{\partial y} g(y,z) \overset{\mathrm{II}}{=} x^2 z$$

$$\Rightarrow \frac{\partial}{\partial y} g(y,z) = 0$$

$$\Rightarrow g(y,z) = h(z)$$

Oppdaterer f:

$f(x,y,z) = x^2 yz + h(z)$

Deriverer mhp z:

$$\frac{\partial f}{\partial z} = x^2 y + \frac{\partial}{\partial z} h(z) \overset{\mathrm{III}}{=} x^2 y + 4z$$

$$\Rightarrow \frac{\partial}{\partial z} h(z) = 4z \Rightarrow h(z) = 2z^2 + C$$

Velger $C = 0$ og får:

$$f(x,y,z) = \underline{\underline{x^2 yz + 2z^2}}$$

Sjekk: $\nabla f = [2xyz,\ x^2 z,\ x^2 y + 4z] = \vec{F}$ ok!

Kurven $\mathbf{r}(t) = (t^2 - 2t)\,\mathbf{i} + e^t\,\mathbf{j} + t^3\mathbf{k}, \qquad 0 \le t \le 2.$

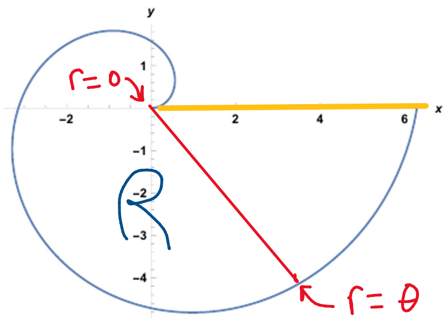
har endepunkter:

$\vec{r}(2) = [0, e^2, 8]$ og $\vec{r}(0) = [0, 1, 0]$

$$\Rightarrow \int_C \vec{F} \cdot d\vec{r} = f(0,e^2,8) - f(0,1,0) = (0 + 2\cdot 8^2) - (0+0) = \underline{\underline{128}}$$

Oppg. 5

a)



$R: \quad 0 \le \theta \le 2\pi$

$\quad\;\; 0 \le r \le \theta$

$$A_R = \iint_R dA = \int_0^{2\pi}\int_0^{\theta} r\,dr\,d\theta$$

$$= \int_0^{2\pi}\left[\tfrac{1}{2}r^2\right]_0^{\theta} d\theta = \tfrac{1}{2}\int_0^{2\pi}\theta^2\,d\theta$$

$$= \tfrac{1}{6}\left[\theta^3\right]_0^{2\pi} = \tfrac{1}{6}\,8\pi^3 = \underline{\underline{\tfrac{4}{3}\pi^3}}$$

b) i) $x = r\cdot\cos\theta = \theta\cdot\cos\theta$

$y = r\cdot\sin\theta = \theta\cdot\sin\theta$

Bruker $t=\theta$ som parameter:

$$\vec{r} = \overbrace{t\cdot\cos t}^{x}\,\underline{i} + \overbrace{t\cdot\sin t}^{y}\,\underline{j}, \quad 0\le t\le 2\pi$$

ii) $$\vec{v} = \frac{d\vec{r}}{dt} = \overbrace{(\cos t - t\cdot\sin t)}^{v_x = \frac{dx}{dt}}\,\underline{i} + \overbrace{(\sin t + t\cdot\cos t)}^{v_y = \frac{dy}{dt}}\,\underline{j}$$

$$|\vec{v}| = \sqrt{v_x^2 + v_y^2}$$

$$= \sqrt{\cos^2 t - \cancel{2t\cos t\cdot\sin t} + t^2\cdot\sin^2 t + \sin^2 t + \cancel{2t\sin t\cdot\cos t} + t^2\cdot\cos^2 t}$$

$$= \sqrt{(\cos^2 t + \sin^2 t)\cdot(1+t^2)} = \sqrt{1+t^2}$$

iii) $$M = \int_C \delta\,ds = \int_0^{2\pi} t\,\overbrace{\sqrt{1+t^2}\,dt}^{ds}$$

$u = 1+t^2, \quad du = 2t\,dt$

$u(2\pi) = 1+4\pi^2, \quad u(0) = 1$

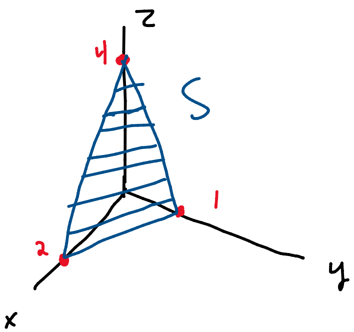
$$= \int_1^{1+4\pi^2} \tfrac{1}{2}u^{\frac{1}{2}}\,du = \tfrac{1}{3}\left[u^{\frac{3}{2}}\right]_1^{1+4\pi^2} = \underline{\underline{\tfrac{1}{3}(1+4\pi^2)^{\frac{3}{2}} - \tfrac{1}{3} \approx 85{,}5}}$$

Oppg. 6

Planet $z = 4 - 2x - 4y$

skjører koordinataksene i:

x	y	z
0	0	4
2	0	0
0	1	0



Merk: området D er en pyramide med grunnflate $G = \frac{2 \cdot 1}{2} = 1$

og volum $V_D = \frac{G \cdot h}{3} = \frac{1 \cdot 4}{3} = \underline{4/3}$

$$\vec{F} = [x, y, z-2]$$

$$\Rightarrow \nabla \cdot \vec{F} = 1 + 1 + 1 = \underline{3}$$

$$\Rightarrow \iiint_D \nabla \cdot \vec{F}\, dV = 3 \cdot \iiint_D dV = 3 \cdot V_D = 3 \cdot \frac{4}{3} = \underline{\underline{4}}$$

Fluks av $\vec{F}$ ut av D gjennom koordinat-planene:

areal til trekant: $\frac{1 \cdot 2}{2} = 1$

$\vec{F}$ er "nesten" radielt!

$$\left[\begin{array}{l} \iint_{xy} \vec{F} \cdot \vec{n}\, d\sigma = \iint_{xy} \overbrace{\underset{0}{[x, y, -2]} \cdot [0,0,-1]}^{2}\, d\sigma = 2 \cdot \overbrace{\iint_{xy} d\sigma}^{1} = \underline{2} \\ \iint_{xz} \vec{F} \cdot \vec{n}\, d\sigma = \iint_{xz} \overbrace{[x, 0, \underset{0}{z-2}] \cdot [0,-1,0]}\, d\sigma = \underline{0} \\ \iint_{yz} \vec{F} \cdot \vec{n}\, d\sigma = \iint_{yz} \overbrace{[0, y, z-2] \cdot [-1,0,0]}\, d\sigma = \underline{0} \end{array}\right.$$

Divergensteoremet gir da:

$$\iint_S \vec{F} \cdot \vec{n}\, d\sigma + \underbrace{\iint_{xy} \vec{F} \cdot \vec{n}\, d\sigma}_{2} + \underbrace{\iint_{xz} \vec{F} \cdot \vec{n}\, d\sigma}_{0} + \underbrace{\iint_{yz} \vec{F} \cdot \vec{n}\, d\sigma}_{0} = \underbrace{\iiint_D \nabla \cdot \vec{F}\, dV}_{4}$$

$$\Rightarrow \iint_S \vec{F} \cdot \vec{n}\, d\sigma = 4 - 2 = \underline{\underline{2}}$$

Oppg. 7 a)

i) $$\frac{d^2x}{dt^2} + a\frac{dx}{dt} + bx + cx^3 = 0$$

m/s²

For at alle leddene skal ha samme enhet (m/s²):

$\dim a = s^{-1}$

$\dim b = s^{-2}$

$\dim c = m^{-2} \cdot s^{-2}$

ii) I) $$\frac{dx}{dt} = v$$

II) $$\frac{dv}{dt} = -av - bx - cx^3$$

iii) Setter $x = L\tilde{x}$, $v = \frac{L}{\tau}\tilde{v}$ og $\frac{dv}{dt} = \frac{L}{\tau^2}\frac{d\tilde{v}}{d\tilde{t}}$

inn i likning II):

$$\frac{L}{\tau^2}\frac{d\tilde{v}}{d\tilde{t}} = -a\frac{L}{\tau}\tilde{v} - bL\tilde{x} - cL^3\tilde{x}^3 \quad \Big| \cdot \frac{\tau^2}{L}$$

$$\frac{d\tilde{v}}{d\tilde{t}} = -\underbrace{a\tau}_{K}\tilde{v} - \underbrace{b\tau^2}_{1}\tilde{x} - \underbrace{cL^2\tau^2}_{1}\tilde{x}^3$$

$$b\tau^2 = 1 \Rightarrow \tau = \underline{\underline{\frac{1}{\sqrt{b}}}}$$

$$cL^2\tau^2 = 1 \Rightarrow L^2 = \frac{1}{c}\cdot\frac{1}{\tau^2} = \frac{b}{c} \Rightarrow L = \underline{\underline{\sqrt{\frac{b}{c}}}}$$

$$K = a\tau = \underline{\underline{\frac{a}{\sqrt{b}}}}$$

Oppg. 7 b)

i) I) $\frac{d\tilde{x}}{d\tilde{t}} = \tilde{v}$

II) $\frac{d\tilde{v}}{d\tilde{t}} = -\tilde{v} - \tilde{x} - \tilde{x}^3$

$$\tilde{x}_{\frac{1}{2}} = \tilde{x}_0 + \left(\frac{d\tilde{x}}{d\tilde{t}}\right)_0 \cdot \frac{\Delta\tilde{t}}{2} = \overset{0.5}{\tilde{x}_0} + \frac{1}{2}\overset{0}{\tilde{v}_0}\cdot\Delta\tilde{t} = \underline{0.5}$$

$$\tilde{v}_{\frac{1}{2}} = \tilde{v}_0 + \left(\frac{d\tilde{v}}{d\tilde{t}}\right)_0 \cdot \frac{\Delta\tilde{t}}{2} = \overset{0}{\tilde{v}_0} - \frac{1}{2}(\overset{0}{\tilde{v}_0} + \overset{\frac{1}{2}}{\tilde{x}_0} + \overset{\frac{1}{8}}{\tilde{x}_0^3})\overset{0.1}{\Delta\tilde{t}}$$

$$= \underline{-0{,}03125}$$

Prøve-steg

$$\tilde{x}_1 = \tilde{x}_0 + \left(\frac{d\tilde{x}}{d\tilde{t}}\right)_{\frac{1}{2}} \cdot \Delta\tilde{t} = \tilde{x}_0 + \tilde{v}_{\frac{1}{2}}\Delta\tilde{t}$$

$$\tilde{x}_1 = 0.5 - 0{,}03125 \cdot 0.1 = \underline{\underline{0{,}49687}}$$

ii) Scriptet implementerer Eulers metode med steglengde $\Delta\tilde{t} = 10^{-3}$ og simulerer fra $t=0$ til $t=0.1$ ($N=100$ iterasjoner).

Eulers metode er en 1.-ordens numerisk metode:

Lokal feil $\sim \Delta\tilde{t}^2 = (10^{-3})^2 = 10^{-6}$

Global feil $\simeq \underset{\text{feil per steg}}{10^{-6}} \cdot \underset{\text{ant. steg}}{10^{2}} = 10^{-4}$

Usikkerheten i tilnærmingen til scriptet, $\tilde{x}(0) = 0{,}4970$, ligger altså i <u>fjerde siffer</u> etter komma.

Midtpunktmetoden er en 2. ordens metode så feilen i tilnærmingen: $\sim \Delta\tilde{t}^3 = (10^{-1})^3 = 10^{-3}$.

Usikkerheten i tilnærmingen $\tilde{x}_1 = 0{,}49687$ ligger i <u>tredje siffer</u> etter komma.

Her gir altså Eulers metode den mest nøyaktige tilnærmingen siden steglengden var mye mindre (100 iterasjoner mot 1 !).