Høgskolen i Østfold

EKSAMEN

Emnekode:	Emnenavn:
IRF30017	Matematikk 3
Dato: 29.11.2019 **Sensurfrist:** 20.12.2019	**Eksamenstid:** 0900-1300
Antall oppgavesider: 3 **Antall vedleggsider:** 7	**Faglærer:** Fredrikstad: Mikjel Thorsrud (41518610) Halden: Einar von Krogh (69608299) **Oppgaven er kontrollert:** Ja

Hjelpemidler:

- Godkjent kalkulator
- Ett A4-ark med valgfritt innhold (maskin eller håndskrevet, kan skrive på begge sider)
- Enten Tor Andersen: "Aktiv formelsamling i matematikk" eller "Gyldendals formelsamling i matematikk"

Om eksamensoppgaven:

Oppgavesettet består av 11 deloppgaver som i utgangspunktet vektes likt:
1, 2, 3, 4a, 4b, 4c, 5a, 5b, 5c, 6a, 6b.
Formelsamling (7 sider) er vedlagt.

Kandidaten må selv kontrollere at oppgavesettet er fullstendig

Oppgave 1 R er området i planet avgrenset av kurvene $y = e^x$, $x = \ln 2$ og koordinataksene. Tegn en skisse og beskriv R i kartesiske koordinater (x, y). Regn ut dobbelintegralet:

$$\iint_R xy \, dA$$

Oppgave 2 Skriv ned ulikhetene som definerer integrasjonsområdet i dobbeltintegralet nedenfor. Tegn en skisse og beskriv integrasjonsområdet i polarkoordinater (r, θ). Regn ut dobbelintegralet.

$$\int_0^3 \int_0^{\sqrt{9-x^2}} \left(x^2 + y^2\right) \, dy \, dx$$

Oppgave 3 I denne oppgaven ser vi på en pyramide med 4 sideflater som alle er trekanter. Pyramiden har hjørner i punktene $O(0, 0, 0)$, $A(2, -2, 0)$, $B(0, 2, 0)$ og $T(0, 0, 2)$. Merk at punktet T er pyramidens topp, mens punktene O, A og B er hjørnene i grunnflaten. Pyramiden har massetetthet $\delta(x, y, z) = 1 + xy^2z$.

i) Vis at $\mathbf{n} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$ er en normalvektor for planet som går gjennom punktene A, B og T. Skriv ned likningen for dette planet.

ii) Tegn en skisse av pyramiden. Tegn også en skisse av pyramidens grunnflate i xy-planet. Skriv ned et beregningsklart trippelintegral som gir pyramidens masse.
NB: Merk at du ikke blir bedt om, og følgelig heller ikke får uttelling for, å regne ut trippelintegralet.

Oppgave 4

a) En kurve C har følgende parametrisering:

$$\mathbf{r} = \cos(\pi t)\,\mathbf{i} \;+\; \sin(\pi t)\,\mathbf{j} \;+\; 2t\,\mathbf{k}\,, \qquad 0 \le t \le 1$$

Bruk parametriseringen til å regne ut linjeintegralet

$$\int_C y\,dx + x\,dy + z^2\,dz$$

b) La K være den rette linjen i rommet som går fra punktet $(1, 0, 0)$ til punktet $(-1, 0, 2)$.

i) Skriv ned en parametrisering for K.

ii) Regn ut linjeintegralet

$$\int_K \mathbf{F} \cdot d\mathbf{r}$$

hvor $\mathbf{F} = y\,\mathbf{i} \;+\; x\,\mathbf{j} \;+\; z^2\,\mathbf{k}$.

c) Regn ut $\nabla \times \mathbf{F}$. Bruk resultatet til å forklare hvorfor linjeintegralene i a) og b) har samme svar.

Oppgave 5 En paraboloide P er definert ved likningen

$$z = 8 - x^2 - 4y^2$$

a) i) Vis at skjæringskurven mellom paraboloiden og xy-planet er en ellipse. Bestem halvaksene a og b til ellipsen. Vis at arealet til ellipsen er $A = 4\pi$ ved å bruke arealformelen $A = \pi ab$.

ii) La R være regionen i xy-planet avgrenset av ellipsen studert over. Forklar at dobbeltintegralet av $f(x, y)$ over R er null hvis funksjonen $f(x, y)$ er odde (antisymmetrisk) i variabelen x. Altså:

$$\iint_R f(x, y)\;dA \;= 0$$

hvis $f(-x, y) = -f(x, y)$ for alle $(x, y) \in R$.
Hint: husk volumtolkningen av dobbeltintegralet.

b) Beregn divergensen og virvlingen til vektorfeltet

$$\mathbf{F} = z\cos x\;\mathbf{i} \;+\; xyz\,\mathbf{j} \;+\; (3 + x^3 z)\;\mathbf{k}$$

c) La flaten S være den delen av paraboloiden P hvor $z \ge 0$. Bruk blant annet divergensteoremet til å regne ut fluksen av $\mathbf{F}$ gjennom S:

$$\iint_S \mathbf{F} \cdot \mathbf{n}\,d\sigma$$

Merk at enhetsnormalen $\mathbf{n}$ ikke er unikt bestemt fordi S ikke er lukket. Du velger selv retning for $\mathbf{n}$; oppgi ditt valg i besvarelsen.
Tips: resultatene fra oppg. a) kan vise seg å være nyttige.

Oppgave 6 En kloss med masse m er festet til en fjær. I denne oppgaven antar vi en kubisk korreksjon til Hookes lov slik at totalkraften på klossen er

$$F_{\text{tot}} = -k_1\, x - k_3\, x^3$$

hvor k_1 og k_3 er positive konstanter og x er klossens posisjon.
(*For spesielt interesserte: Hookes lov er ikke annet enn Taylor-rekken avbrutt etter det lineære leddet. En kvadratisk korreksjon ville bryte symmetrien om likevektspunktet* $x = 0$, *som er grunnen til at vi her antar at den ledende korreksjonen er kubisk.*)

a) i) Skriv ned enhetene til fjær-konstantene k_1 og k_3 i SI-systemet.

ii) Skriv ned en andre-ordens differensiallikning for posisjonen $x(t)$ ved å bruke Newtons andre lov.

iii) Vi innfører dimensjonsløse variable $\tilde{t}$ og $\tilde{x}$ definert

$$t = \tau \cdot \tilde{t}, \quad x = L \cdot \tilde{x},$$

hvor τ er en tidsskala med SI-enhet s (sekund) og L er en lengdeskala med SI-enhet m (meter). Vis at differensiallikningen kan skrives på dimensjonsløs form som

$$\frac{d^2\tilde{x}}{d\tilde{t}^2} + \tilde{x} + \tilde{x}^3 = 0$$

ved passende valg av τ og L (uttrykt ved k_1, k_3 og m).

iv) Kontroller at uttrykket ditt for τ har enhet sekund (s) og at uttrykket ditt for L har enhet meter (m).
(*Kommentar: Dette er eksempel på en "konsistenssjekk", en sjekk av at utregningen ikke førte til selvmotsigelser.)*

b) Ved tidspunktet $\tilde{t} = 0$ er posisjonen $\tilde{x}_0 = 1$ og hastigheten $\tilde{v}_0 = 1$. Bruk midtpunktmetoden til å finne en tilnærmet verdi for posisjonen $\tilde{x}$ ved tidspunktet $\tilde{t} = 0.1$. Bruk kun ett tidssteg, dvs. regn ut $\tilde{x}_1$ ved å bruke tidssteget $\Delta\tilde{t} = 0.1$. Gi en kort kommentar til nøyaktigheten til tilnærmingen med utgangspunkt i den numeriske metoden du har brukt.
Tips: begynn som alltid med å skrive differensiallikningen om til 2 koblede første-ordens differensiallikninger.

Collection of formulas – Matematikk 3 (IRF30017)

Conic sections

Conic sections on standard form with foci on the x-axis:

$$\begin{aligned}
&\text{Ellipse:} && \frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \quad a>b, \quad \text{foci:} \quad (\pm c,0), \quad c=\sqrt{a^2-b^2}.\\
&\text{Hyperbola:} && \frac{x^2}{a^2}-\frac{y^2}{b^2}=1, \quad \text{foci:} \quad (\pm c,0), \quad c=\sqrt{a^2+b^2}, \quad \text{asymptotes}: \quad y=\pm(b/a)x.\\
&\text{Parabola:} && y=\frac{x^2}{4p}, \quad \text{focus:} \quad (0,p), \quad \text{directrix:} \quad y=-p.
\end{aligned}$$

In the case of the ellipse, a is called the *semimajor axis* and b the *semiminor axis.*

English – norwegian: conic section – kjeglesnitt, directrix – styrelinje, focus – fokus eller brennpunkt, semimajor axis – store halvakse, semiminor axis – lille halvakse.

The method of Lagrange multipliers

Assume that $f(x_1,\ldots,x_n)$ and $g(x_1,\ldots,x_n)$ are differentiable functions and that $\nabla g \neq 0$ when $g=0$. The stationary points of f subject to the constraint $g=0$ are found by solving the $n+1$ scalar equations

$$\nabla f = \lambda \nabla g, \quad g=0$$

for the $n+1$ unknowns $\lambda, x_1,\ldots,x_n$. The stationary points are *candidates* for local maxima and minima of f subject to $g=0$.

Double and triple integrals

Cartesian (x,y,z), cylindrical (r,θ,z) and spherical (ρ,ϕ,θ) coordinates of a point P:

$$\begin{aligned}
\text{From cylindrical to Cartesian:} &\quad x=r\cos\theta,\ y=r\sin\theta,\ z=z.\\
\text{From spherical to cylindrical:} &\quad r=\rho\sin\phi,\ \theta=\theta,\ z=\rho\cos\phi.\\
\text{From spherical to Cartesian:} &\quad x=\rho\sin\phi\cos\theta,\ y=\rho\sin\phi\sin\theta,\ z=\rho\cos\phi.
\end{aligned}$$

$r=\sqrt{x^2+y^2}$ is the distance to the z axis and $\rho=\sqrt{x^2+y^2+z^2}$ is the distance to the origin ($|\overrightarrow{OP}|$).
$\theta \in [0,2\pi]$ is the polar angular coordinate of the projection of P on the xy-plane and $\phi \in [0,\pi]$ is the angle between the z-axis and $\overrightarrow{OP}$.

Area and volume elements:

$$\begin{aligned}
&dA = dx\,dy = r\ dr\,d\theta = |J(u,v)|\ du\,dv,\\
&dV = dx\,dy\,dz = r\ dz\,dr\,d\theta = \rho^2\sin\phi\ d\rho\,d\phi\,d\theta = |J(u,v,w)|\ du\,dv\,dw,\\
&J(u,v)=\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}, \quad J(u,v,w)=\frac{\partial(x,y,z)}{\partial(u,v,w)}
\end{aligned}$$

Applications of double and triple integrals:

$$\begin{aligned}
&\text{Area of } R: \quad A=\iint_R dA, && \text{Volume of } D: \quad V=\iiint_D dV\\
&\text{Average of } f \text{ over } R: \quad \overline{f}=\frac{1}{A}\iint_R f(x,y)\,dA, && \text{Average of } f \text{ over } D: \quad \overline{f}=\frac{1}{V}\iiint_D f(x,y,z)\,dV
\end{aligned}$$

Object with mass density $\delta(x,y,z)$ occupying a region D in space:

$$\text{Mass:} \quad M=\iiint_D \delta(x,y,z)\,dV, \qquad \text{Center of mass:} \quad \overline{x}=\frac{M_{yz}}{M}, \quad \overline{y}=\frac{M_{xz}}{M}, \quad \overline{z}=\frac{M_{xy}}{M},$$

$$M_{yz}=\iiint_D x\,\delta(x,y,z)\,dV, \quad M_{xz}=\iiint_D y\,\delta(x,y,z)\,dV, \quad M_{xy}=\iiint_D z\,\delta(x,y,z)\,dV$$

Parametric curves and line integrals

Below the following parametrization of a curve C in space is assumed:

$$C: \quad \mathbf{r}(t) = g(t)\mathbf{i} + h(t)\mathbf{j} + k(t)\mathbf{k}, \qquad a \le t \le b$$

$$\text{Tangent vector:} \quad \mathbf{v}(t) = \frac{d\mathbf{r}}{dt} = g'(t)\mathbf{i} + h'(t)\mathbf{j} + k'(t)\mathbf{k}, \quad \text{Unit tangent vector:} \quad \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|}, \quad |\mathbf{v}| = \sqrt{\mathbf{v}\cdot\mathbf{v}},$$

$$\text{Arc length:} \quad L = \int_a^b |\mathbf{v}|\,dt, \qquad \text{Arc length parameter:} \quad s(t) = \int_a^t |\mathbf{v}(t')|dt'$$

Relations between differentials:

$$d\mathbf{r} = dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}, \quad d\mathbf{r} = \mathbf{T}ds, \quad ds = |\mathbf{v}|dt$$

Line integral of scalar $f(x, y, z)$ along C:

$$\int_C f(x,y,z)\,ds = \int_a^b f(\mathbf{r}(t))\,|\mathbf{v}(t)|\,dt, \quad f(\mathbf{r}(t)) = f(g(t), h(t), k(t))$$

Line integral of vector field $\mathbf{F}(x, y, z) = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + P(x, y, z)\mathbf{k}$ along C:

$$\overbrace{\int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_C M dx + N dy + P dz}^{\text{definitions}} = \overbrace{\int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{v}\,dt}^{\text{how to calculate}}$$

The line integral of the x-component of $\mathbf{F}$ along C:

$$\int_C M(x,y,z)\,dx = \int_a^b M(\mathbf{r}(t))\frac{dx}{dt}\,dt = \int_a^b M(g(t), h(t), k(t))\,g'(t)\,dt$$

English – norwegian: line integral – linjeintegral, unit tangent vektor – enhets-tangentvektor, arc length – buelengde.

Names on line integrals: work, flow, circulation and flux

Let $\mathbf{F}$ be a vector field in $\mathbb{R}^n$ and C a parametrized curve in the same space. The line integral

$$\int_C \mathbf{F}\cdot d\mathbf{r}$$

is called the

- *work* done by $\mathbf{F}$ on an object moving along the curve C if $\mathbf{F}$ is a force field
- *flow* of $\mathbf{F}$ along C if $\mathbf{F}$ is a velocity field
- *circulation* of $\mathbf{F}$ along C if $\mathbf{F}$ is a velocity field and C is a closed curve (for a closed curve the line integral is often written $\oint_C$)

Flux integral in two dimensions: Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ be a vector field and C a simple closed curve in the plane ($\mathbb{R}^2$) with unit normal $\mathbf{n}$ oriented outwards. The following line integral is the flux of $\mathbf{F}$ across the curve C:

$$\text{flux} = \oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \oint_C M\,dy - N\,dx$$

Flux integral in three dimensions: see surface integrals below.

English – norwegian: work – arbeid, flow – strøm, circulation – sirkulasjon, flux – fluks.

del, divergence and curl

Del operator:

$$\mathbb{R}^3: \quad \nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}, \qquad \mathbb{R}^n: \quad \nabla = \sum_{i=1}^{n} \mathbf{e}_i \frac{\partial}{\partial x^i}$$

The following definitions assume that $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is a vector field in space ($\mathbb{R}^3$), but the divergence generalizes naturally to a space of arbitrary dimensions ($\mathbb{R}^n$):

$$\text{Divergence of } \mathbf{F}: \quad \operatorname{div}\mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}, \qquad \text{Curl of } \mathbf{F}: \quad \operatorname{curl}\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix}$$

Identities: $\nabla \times (\nabla f) = 0, \quad \nabla \cdot (\nabla \times \mathbf{F}) = 0$
English – norwegian: del – nabla, divergence – divergens, curl – virvling.

Conservative fields and path independence

The following statements are equivalent if $\mathbf{F}$ is a vector field in space whose components have continuous partial derivatives in a connected and simply connected domain D and C is a curve in the same domain:

1. $\mathbf{F}$ is conservative
   (this is another way to say that the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is path independent)
2. $\mathbf{F}$ is curl-free, $\nabla \times \mathbf{F} = \vec{\mathbf{0}}$
   (this provides a component test for conservative fields, in the plane write $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + 0\mathbf{k}$)
3. $\mathbf{F}$ is a gradient field: $\mathbf{F} = \nabla f$
   (the function $f(x, y, z)$ is called a *potential function* for $\mathbf{F}$)
4. $\int_A^B \mathbf{F} \cdot d\mathbf{r} = f(B) - f(A)$ for all curves C from A to B
5. $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ for all closed curves C

English – norwegian: conservative – konservativ, path independent – veiuavhengig, curl-free – virvelfri.

Green's theorem

Let R be a region in the plane bounded by the piecewise smooth, simple closed curve C and let $\mathbf{F} = M(x, y)\mathbf{i} + N(x, y)\mathbf{j} + 0\mathbf{k}$ be a vector field with components M and N that have continuous partial derivatives.
Circulation-curl form:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \nabla \times \mathbf{F} \cdot \mathbf{k}\, dA$$

or

$$\oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\, dy$$

Flux-divergence form:

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_R \nabla \cdot \mathbf{F}\, dA$$

or

$$\oint_C M dy - N dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dx\, dy$$

English – norwegian: piecewise smooth – stykkevis glatt, simple curve – enkel kurve.

Surface integrals

Let S be a smooth surface in space ($\mathbb{R}^3$). The area element $d\sigma$ depends on the description of S:

1) $d\sigma = |\mathbf{r}_u \times \mathbf{r}_v| \, du \, dv$ if S is given **parametrically** as $\mathbf{r}(u,v) = f_1(u,v)\mathbf{i} + f_2(u,v)\mathbf{j} + f_3(u,v)\mathbf{k}$

2) $d\sigma = \frac{|\nabla G|}{|\nabla G \cdot \mathbf{k}|} \, dx \, dy$ if S is given **implicitly** by the equation $G(x,y,z) = 0$

3) $d\sigma = \sqrt{g_x^2 + g_y^2 + 1} \, dx \, dy$ if S is given **explicitly** as the the graph $z = g(x,y)$

Below the case 3) of an explicitly defined surface is assumed. Let R be the shaddow of S on the xy-plane. The area of S is:

$$A = \iint_S d\sigma = \iint_R \sqrt{g_x^2 + g_y^2 + 1} \, dx \, dy$$

The integral of a scalar $f(x,y,z)$ over S:

$$\iint_S f(x,y,z) d\sigma = \iint_R f(x,y,g(x,y)) \sqrt{g_x^2 + g_y^2 + 1} \, dx \, dy$$

A surface has two unit normal fields:

$$\mathbf{n} = \pm \frac{\nabla G}{|\nabla G|} = \pm \frac{-g_x \mathbf{i} - g_y \mathbf{j} + \mathbf{k}}{\sqrt{g_x^2 + g_y^2 + 1}}$$

For a given choice of $\mathbf{n}$ the *flux* of $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ across S is:

$$\text{Flux} = \iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = \operatorname{sgn}(\mathbf{n} \cdot \mathbf{k}) \iint_R -M g_x - N g_y + P \, dx \, dy$$

English – norwegian: surface integral – flateintegral, unit normal – enhetsnormal.

Stoke's theorem and the divergence theorem

Let S be an oriented piecewise smoooth surface in space having a piecewise smooth boundrary curve C that is directed counterclockwise relative to the unit normal field $\mathbf{n}$ of S. Stokes theorem:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} \, d\sigma$$

Let D be a region in space with a piecewise smooth boundrary surface S having an outward unit normal field $\mathbf{n}$. Divergence theorem:

$$\oiint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = \iiint_D \nabla \cdot \mathbf{F} \, dV$$

In both thorems the components of $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ have continuous partial derivatives.

English – norwegian: boundrary – rand, boundrary curve – randkurve, boundrary surface – randflate, outward unit normal – utoverrettet enhetsnormal.

Modeling in physics

Numerical methods

Consider the first-order differential equation:

$$\frac{du}{dt} = f(u,t)$$

Let u_n be a numerical approximation to $u(t_n)$, where $t_n = t_0 + n\Delta t$.

Euler method: Use the tangent at the previous point to estimate the next:

$$u_{n+1} = u_n + \left(\frac{du}{dt}\right)_n \Delta t = u_n + f(u_n, t_n)\Delta t$$

Or more compactly:

$$u_{n+1} = u_n + f_n \Delta t$$

First order method (local error: $\sim \Delta t^2$, global error: $\sim \Delta t$).

Midpoint method: Use Euler's method with a half time step to estimate the slope at the midpoint (trial step), then apply this to estimate the next point:

$$\begin{aligned} u_{n+\frac{1}{2}} &= u_n + \left(\frac{du}{dt}\right)_n \frac{\Delta t}{2} = u_n + \frac{1}{2} f(u_n, t_n)\Delta t, \quad \text{(trial step)}, \\ u_{n+1} &= u_n + \left(\frac{du}{dt}\right)_{n+\frac{1}{2}} \Delta t = u_n + f(u_{n+\frac{1}{2}}, t_n + \frac{\Delta t}{2})\Delta t \end{aligned}$$

Or more compactly:

$$\begin{aligned} u_{n+\frac{1}{2}} &= u_n + \frac{1}{2} f_n \Delta t, \quad \text{(trial step)}, \\ u_{n+1} &= u_n + f_{n+\frac{1}{2}} \Delta t \end{aligned}$$

Second order method (local error: $\sim \Delta t^3$, global error: $\sim \Delta t^2$).

Higher order differential equations

A second order differential equation can be rewritten as a system of two coupled first order equations:

$$\frac{d^2u}{dt^2} = f\left(u, \frac{du}{dt}, t\right) \quad \Longleftrightarrow \quad \begin{aligned} &\textbf{I.} \quad \frac{du}{dt} = v, \\ &\textbf{II.} \quad \frac{dv}{dt} = f\left(u, v, t\right) \end{aligned}$$

The numerical schemes above can then be applied to find u_{n+1} and v_{n+1} from u_n and v_n.

Dimensionless variables

An ordinary differential equation for $x(t)$ can be written on dimensionless form by introducing a length scale L and time scale τ:

$$x = L\tilde{x}, \quad t = \tau\tilde{t} \quad \longrightarrow \quad \frac{dx}{dt} = \frac{L}{\tau}\frac{d\tilde{x}}{d\tilde{t}}, \quad \frac{d^2x}{dt^2} = \frac{L}{\tau^2}\frac{d^2\tilde{x}}{d\tilde{t}^2},$$

where in SI units $\text{Dim}(x) = \text{Dim}(L) = \text{m}$, $\text{Dim}(t) = \text{Dim}(\tau) = \text{s}$ and $\text{Dim}(\tilde{x}) = \text{Dim}(\tilde{t}) = 1$.
SI base units: m, s, kg. SI derived units: N=kg·ms^{-2} (Newton's 2nd law), J=N·m (work-energy theorem).

Some solutions of selected differential equations

Harmonic oscillator equation (ordinary, linear, homogeneous):

$$\frac{d^2x}{dt^2} + w^2 x = 0 \quad \rightarrow \quad x(t) = A\cos(wt + \phi)$$

Amplitude: A [m], angular frequency: w [rad/s], frequency: $f = \frac{w}{2\pi}$ [Hz], period: $T = \frac{1}{f} = \frac{2\pi}{w}$, phase: ϕ [rad].

One-dimensional wave equation (partial, linear, homogeneous):

$$\frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2}$$

Mechanical waves on a string:

- Harmonic wave travelling to the right: $y(x,t) = A\cos(kx - wt + \phi), \quad w = v \cdot k$.
  Wave number: k [m^{-1}], wave length: $\lambda = \frac{2\pi}{k}$ [m].
- Standing waves with boundrary conditions $y(0,t) = y(L,t) = 0$:
  $y(x,t) = A\sin(kx) \cdot \cos(wt), \quad w = v \cdot k, \quad k = \frac{n\pi}{L}, \quad n = 1, 2, 3, \ldots$

One-dimensional heat equation / diffusion equation (partial, linear, homogeneous):

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \rightarrow \quad u(x,t) = A\sin(kx) \cdot e^{-(ck)^2 t}, \quad k = \frac{n\pi}{L}, \quad n = 1, 2, 3, \ldots$$

The solutions above satisfy the boundrary conditions $u(0,t) = u(L,t) = 0$.

From previous courses

Scalar product and vector product

When $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$:

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| \cdot |\mathbf{b}| \cos\alpha = a_1 b_1 + a_2 b_2 + a_3 b_3, \qquad \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}, \quad |\mathbf{a} \times \mathbf{b}| = |\mathbf{a}| \cdot |\mathbf{b}| \sin\alpha$$

Straight line in space

Parametrization of a line through the point $P_0(x_0, y_0, z_0)$ parallell to $\vec{v} = [a, b, c]$:

$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v} = (x_0 + at)\mathbf{i} + (y_0 + bt)\mathbf{j} + (z_0 + ct)\mathbf{k}, \quad -\infty \le t \le \infty$$

A possible parametrization of a straight line from $\mathbf{r}_1$ to $\mathbf{r}_2$:

$$\mathbf{r}(t) = \mathbf{r}_1 + (\mathbf{r}_2 - \mathbf{r}_1)t, \quad 0 \le t \le 1$$

Plane in space

Equation for a plane through the point $P_0(x_0, y_0, z_0)$ normal to $\vec{n} = [a, b, c]$:

$$\overrightarrow{P_0 P} \cdot \vec{n} = 0 \quad \to \quad (x - x_0)a + (y - y_0)b + (z - z_0)c = 0$$

Circle in the plane

Equation for a circle with radius a and center in (x_0, y_0): $(x - x_0)^2 + (y - y_0)^2 = a^2$

Taylor expansion

Taylor series of a function $f(x)$ about the point $x = a$:

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x - a)^k = f(a) + f'(a)(x - a) + \frac{1}{2!} f''(a)(x - a)^2 + \dots$$

Taylor polynom of degree n:

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x - a)^k = f(a) + f'(a)(x - a) + \frac{1}{2!} f''(a)(x - a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x - a)^n$$

Linear approximation to $f(x)$ around $x = a$:

$$f(x) \simeq f(a) + f'(a)(x - a) \quad \text{if} \quad \left| \frac{1}{2} f''(a)(x - a)^2 \right| \ll |f'(a)(x - a)|$$

Some trigonometric identities

$$\sin^2 u + \cos^2 u = 1, \quad \sin(u + v) = \sin u \, \cos v + \cos u \, \sin v, \quad \cos(u + v) = \cos u \, \cos v - \sin u \, \sin v,$$
$$\sin(2u) = 2 \sin u \cos u, \quad \cos(2u) = \cos^2 u - \sin^2 u, \quad \cos^2 u = (1 + \cos(2u))/2, \quad \sin^2 u = (1 - \cos(2u))/2$$