LF Matematikk 3, april 2019 (Konte)

1 a)

$$\int_0^{2\pi}\int_0^{\pi/2}\int_0^{3} \rho^2 \cdot \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/2} \left[\frac{1}{3}\rho^3 \cdot \sin\phi\right]_{\rho=0}^{\rho=3} d\phi \, d\theta$$

$$(-0-(-1)) = 1$$

$$= 9\int_0^{2\pi}\int_0^{\pi/2} \sin\phi \, d\phi \, d\theta = 9 \cdot \int_0^{2\pi} \left[-\cos\phi\right]_0^{\pi/2} d\theta$$

$$= 9\int_0^{2\pi} d\theta = \underline{\underline{18\pi}}$$

Integrasjonsområdet i sfæriske koordinater:

$0 \le \theta \le 2\pi$

$0 \le \phi \le \pi/2$

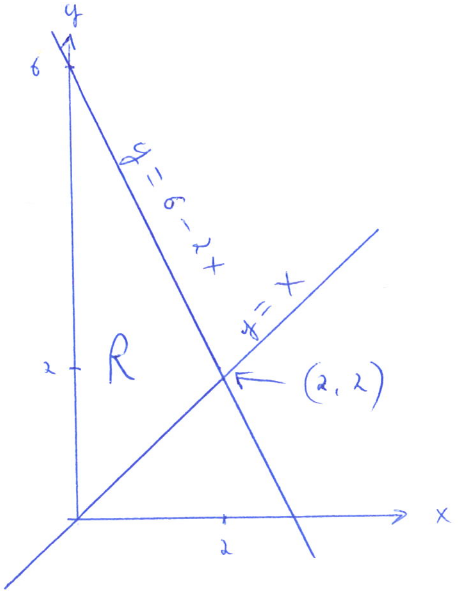
$0 \le \rho \le 3$

$\longrightarrow$ Halvkule med radius 3.

Sjekk:

$$V = \frac{1}{2} \cdot \frac{4}{3}\pi r^3 = 18\pi$$

1 (b)



$R: \quad 0 \le x \le 2$
$\quad\; x \le y \le 6-2x$

$$\iint_R 3xy \, dA = \int_0^2 \int_x^{6-2x} 3xy \, dy \, dx$$

$$= \int_0^2 \left[\frac{3}{2} x y^2\right]_{y=x}^{y=6-2x} dx$$

$$= \int_0^2 \frac{3}{2} x \left[(6-2x)^2 - x^2\right] dx$$

$$= \int_0^2 \frac{3}{2} x \left(36 - 24x + 4x^2 - x^2\right) dx$$

$$= \int_0^2 \frac{3}{2} x \left(36 - 24x + 3x^2\right) dx$$

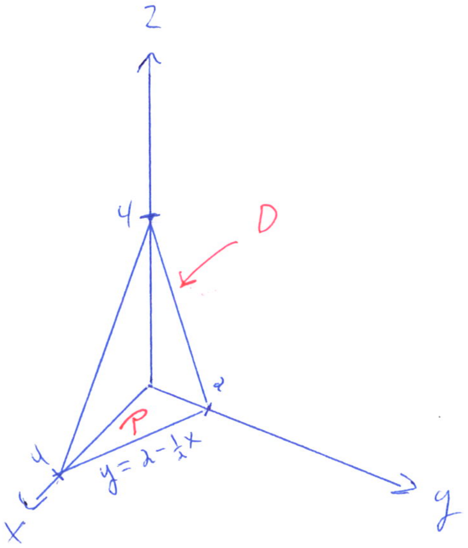
$$= \int_0^2 54x - 36x^2 + \frac{9}{2} x^3 \, dx$$

$$= \left[27x^2 - 12x^3 + \frac{9}{8} x^4\right]_0^2$$

$$= \left(27 \cdot 4 - 12 \cdot 8 + \frac{9}{8} \cdot 16\right) - 0 = \underline{\underline{30}}$$

1 c)

i)



$$D: \quad \left.\begin{array}{l} 0 \le x \le 4 \\ 0 \le y \le 2 - \frac{1}{2}x \end{array}\right] R$$
$$0 \le z \le 4 - x - 2y$$

Volum:

$$V = \iiint_D dV = \int_0^4 \int_0^{2-\frac{1}{2}x} \int_0^{4-x-2y} dz\,dy\,dx = \frac{16}{3}$$

eller ved volum formel for pyramide:

$$V = \frac{G \cdot h}{3} = \frac{\frac{4 \cdot 2}{2} \cdot 4}{3} = \underline{\underline{\frac{16}{3}}}$$

ii) $$\overline{T} = \frac{1}{V} \cdot \iiint_D T(x,y,z)\,dV$$

$$= \frac{3}{16} \cdot \int_0^4 \int_0^{2-\frac{1}{2}x} \int_0^{4-x-2y} (20+z)\,dz\,dy\,dx$$

$$= 21$$

2 a) Standard form:

$$\frac{x^2}{4^2} + \frac{y^2}{2^2} = 1$$

Store halvakse: $a = 4$.
Lille halvakse: $b = 2$.

Liggende ellipse med brennpunktene på x-aksen i punktene $(\pm c, 0)$, hvor

$$c = \sqrt{a^2 - b^2} = 2\sqrt{3}$$

2 b)

i) $$\vec{r} = \overbrace{4 \cdot \cos(\pi t)}^{x}\, \underline{i} + \overbrace{2 \cdot \sin(\pi t)}^{y}\, \underline{j}$$

Innsatt på venstre side i ligning for ellipsen:

$$v.s = x^2 + 4y^2 = (4 \cdot \cos(\pi t))^2 + 4 \cdot (2 \cdot \sin(\pi t))^2$$

$$= 16\,(\underbrace{\cos^2(\pi t) + \sin^2(\pi t)}_{=1})$$

$$= 16 = h.s$$

som viser at alle punktene ligger på ellipsen. C dekker hele ellipsen siden $\pi t \in [0, 2\pi]$.

ii) $$\vec{r}\left(\tfrac{1}{6}\right) = 4 \cdot \cos\frac{\pi}{6}\, \underline{i} + 2 \cdot \sin\frac{\pi}{6}\, \underline{j} = 2\sqrt{3}\, \underline{i} + 1\, \underline{j}$$

$$\Rightarrow \underline{\underline{A(\overset{x}{2\sqrt{3}}, \overset{y}{1})}}$$

2 c) i)

$$\vec{r}(t) = \left[4 \cdot \cos \pi t \,,\; 2 \cdot \sin \pi t\right]$$

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = \left[-4\pi \cdot \sin \pi t \,,\; 2\pi \cdot \cos \pi t\right]$$

$$\vec{v}\left(\tfrac{1}{6}\right) = \left[-4\pi \cdot \sin \frac{\pi}{6} \,,\; 2\pi \cdot \cos \frac{\pi}{6}\right]$$

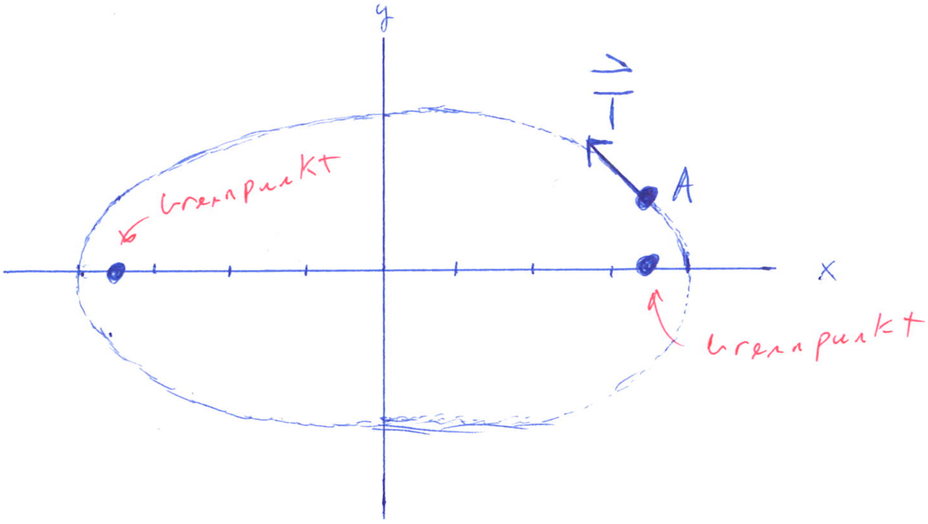
$$\vec{v}\left(\tfrac{1}{6}\right) = \left[-2\pi \,,\; \sqrt{3}\,\pi\right]$$

$$\left|\vec{v}\left(\tfrac{1}{6}\right)\right| = \sqrt{(-2\pi)^2 + (\sqrt{3}\pi)^2} = \sqrt{7}\,\pi$$

$$\vec{T}\left(\tfrac{1}{6}\right) = \frac{\vec{v}\left(\frac{1}{6}\right)}{\left|\vec{v}\left(\frac{1}{6}\right)\right|} = \underline{\underline{\left[-\frac{2}{\sqrt{7}} \,,\; \sqrt{\frac{3}{7}}\right]}}$$

Med desimaltall: $\vec{T}\left(\tfrac{1}{6}\right) \simeq \left[-0{,}76 \,,\; 0{,}65\right]$

ii)



d) $\vec{F} = \left[-y\cdot e^{2z},\ x+yz,\ z\right]$

$$\operatorname{div}\vec{F} = \nabla\cdot\vec{F}$$

$$= \frac{\partial}{\partial x}\left(-y\cdot e^{2z}\right) + \frac{\partial}{\partial y}\left(x+yz\right) + \frac{\partial z}{\partial z}$$

$$= 0 + z + 1$$

$$= \underline{\underline{z+1}}$$

$$\operatorname{curl}\vec{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial_x & \partial_y & \partial_z \\ -y\cdot e^{2z} & x+yz & z \end{vmatrix}$$

$$= \underline{i}\left[\frac{\partial z}{\partial y} - \frac{\partial}{\partial z}(x+yz)\right]$$

$$- \underline{j}\left[\frac{\partial z}{\partial x} - \frac{\partial}{\partial z}\left(-y\cdot e^{2z}\right)\right]$$

$$+ \underline{k}\left[\frac{\partial}{\partial x}(x+yz) - \frac{\partial}{\partial y}\left(-y\cdot e^{2z}\right)\right]$$

$$= \underline{\underline{-y\,\underline{i} - 2y\cdot e^{2z}\,\underline{j} + 1 + e^{2z}\,\underline{k}}}$$

$\nabla\times\vec{F} \neq \vec{0} \implies \vec{F}$ er <u>ikke</u> konservativt.

e) S er definert

$$\frac{x^2}{4^2} + \frac{y^2}{2^2} + \frac{z^2}{3^2} = 1 , \quad z \geq 0.$$

Randkurven til S får vi ved å sette $z = 0$ som gir ellipsen med parametrisert kurve C studert i oppgavene over.

Stoke's theorem:

$$\iint_S \nabla \times \vec{F} \cdot \vec{n} \, d\sigma = \oint_C \vec{F} \cdot d\vec{r}$$

I planet $z = 0$ er vektorfeltet

$$\vec{F}(x, y, 0) = [-y , x , 0]$$

Som evaluert langs randkurven

$$\vec{r}(t) = \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left[4 \cdot \cos \pi t , \; 2 \cdot \sin \pi t , \; 0 \right] \end{array}$$ Kjent fra b)

blir

$$\vec{F}(t) = [-2 \sin \pi t , \; 4 \cdot \cos \pi t , \; 0]$$

Tangent vektoren til kurven:

$$\vec{v} = \frac{d\vec{r}}{dt} = [-4\pi \sin \pi t , \; 2\pi \cos \pi t , \; 0]$$

$$\vec{F}(t) \cdot \vec{v}(t) = 8\pi \cdot \sin^2(\pi t) + 8\pi \cdot \cos^2 \pi t$$

$$= 8\pi$$

Vi får dermed:

$$\iint_S \nabla \times \vec{F} \cdot \vec{n} \, d\sigma = \oint_C \vec{F} \cdot d\vec{r}$$

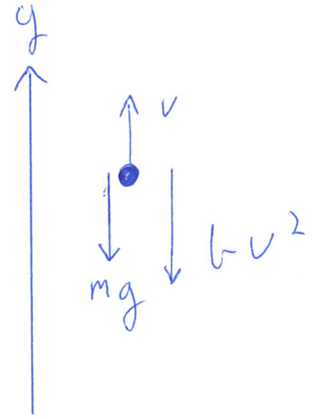
$$= \int_0^2 \vec{F} \cdot \frac{d\vec{r}}{dt} \, dt$$

$$= \int_0^2 8\pi \, dt$$

$$= \underline{\underline{16\pi}}$$

3 a)

i)



Newton's 2. lov med positiv retning oppover:

$$\Sigma F = m \cdot a$$

$$-bv^2 - mg = m \cdot \frac{dv}{dt}$$

$$\frac{d^2y}{dt^2} + \frac{b}{m}\left(\frac{dy}{dt}\right)^2 + g = 0$$

ii) Setter $\frac{dy}{dt} = \frac{L}{\tau}\frac{d\tilde{y}}{d\tilde{t}}$ og $\frac{d^2y}{dt^2} = \frac{L}{\tau^2}\frac{d^2\tilde{y}}{d\tilde{t}^2}$:

$$\frac{L}{\tau^2}\frac{d^2\tilde{y}}{d\tilde{t}^2} + \frac{b}{m}\frac{L^2}{\tau^2}\left(\frac{d\tilde{y}}{d\tilde{t}}\right)^2 + g = 0$$

$$\frac{d^2\tilde{y}}{d\tilde{t}^2} + \overbrace{\frac{b \cdot L}{m}}^{1}\left(\frac{d\tilde{y}}{d\tilde{t}}\right)^2 + \overbrace{\frac{g\tau^2}{L}}^{1} = 0$$

Krav 1: $\frac{b \cdot L}{m} = 1 \Rightarrow \underline{\underline{L = \frac{m}{b}}}$

Krav 2: $\frac{g\tau^2}{L} = 1 \Rightarrow \tau = \sqrt{\frac{L}{g}} = \underline{\underline{\sqrt{\frac{m}{gb}}}}$

$$\Rightarrow \frac{d^2\tilde{y}}{d\tilde{t}^2} + \left(\frac{d\tilde{y}}{d\tilde{t}}\right)^2 + 1 = 0$$

b) I. $\frac{d\tilde{y}}{d\tilde{t}} = \tilde{v}$

II. $\frac{d\tilde{v}}{d\tilde{t}} = -\tilde{v}^2 - 1$

<u>Prøvesteg:</u>

$$\tilde{y}_{\frac{1}{2}} = \tilde{y}_0 + \left(\frac{d\tilde{y}}{d\tilde{t}}\right)_0 \cdot \frac{\Delta\tilde{t}}{2}$$

$$= \overset{=1}{\tilde{y}_0} + \frac{1}{2}\overset{=1}{\tilde{v}_0} \cdot \Delta\overset{=0.1}{\tilde{t}} = \underline{1,05}$$

$$\tilde{v}_{\frac{1}{2}} = \tilde{v}_0 + \left(\frac{d\tilde{v}}{d\tilde{t}}\right)_0 \cdot \frac{\Delta\tilde{t}}{2}$$

$$= \tilde{v}_0 + \left(-\tilde{v}^2 - 1\right)_0 \cdot \frac{\Delta\tilde{t}}{2}$$

$$= \overset{=1}{\tilde{v}_0} - \frac{1}{2}\left(\overset{=1}{\tilde{v}^2} + 1\right) \Delta\overset{=0.1}{\tilde{t}} = \underline{0,9}$$

<u>Neste steg:</u>

$$\tilde{y}_1 = \tilde{y}_0 + \left(\frac{d\tilde{y}}{d\tilde{t}}\right)_{\frac{1}{2}} \cdot \Delta\tilde{t}$$

$$= \overset{=1}{\tilde{y}_0} + \overset{=0.9}{\tilde{v}_{\frac{1}{2}}} \cdot \Delta\overset{=0.1}{\tilde{t}}$$

$$= \underline{\underline{1,09}}$$

Posisjonen er ca $\tilde{y} = 1,09$ ved $\tilde{t} = 0.1$.

c) $\tilde{y}(\tilde{t}) = A + \ln\left[\cos(\tilde{t} - B)\right]$

$$\frac{d\tilde{y}}{d\tilde{t}} = 0 + \frac{1}{\cos(\tilde{t}-B)} \cdot \left(-\sin(\tilde{t}-B)\right)$$

$$= -\tan(\tilde{t} - B)$$

Initialbetingelsene $\tilde{y}(0) = \tilde{v}(0) = 1$ gir

1) $A + \ln\left[\cos(-B)\right] = 1$

2) $-\tan(-B) = 1$

eller ved å bruke at $\cos(x) = \cos(-x)$

og $\tan(x) = -\tan(-x)$:

1) $A + \ln\left[\cos B\right] = 1$

2) $\tan B = 1$

Fra 2) får vi:

$$\underline{\underline{B = \frac{\pi}{4}}}$$

som innsatt i 1) gir

$$A = 1 - \ln \cos \frac{\pi}{4}$$

$$= \underline{\underline{1 - \ln \frac{\sqrt{2}}{2}}}$$

Den eksakte posisjonen blir da:

$$\tilde{y}(0.1) = A + \ln\left[\cos(0{,}1 - B)\right]$$

$$= \underline{1{,}09061}$$

Feilen i tilnærmingen blir:

$$|\tilde{y}(0.1) - \tilde{y}_1| = |1{,}09061 - 1{,}09000|$$

$$= 0{,}00061$$

$$\simeq 10^{-3}$$

Feilen er av størrelsesorden $\sim \Delta\tilde{t}^3 = 10^{-3}$ per tidssteg $\Delta\tilde{t}$

som passer med at midtpunktmetoden er en andre-ordens metode, altså nøyaktig til orden $\Delta\tilde{t}^2$.

(NB: her var alle koeffisienter og initialbetingelser lik 1. Generelt må vi se sammenligne Taylor-rekkene.)