EKSAMEN

Emnekode:	Emnenavn:
IRF30017	Matematikk 3
Dato: 07.01.2019 **Sensurfrist:** 28.01.2019	**Eksamenstid:** 0900-1300
Antall oppgavesider: 3 **Antall vedleggsider:** 7	**Faglærer:** Fredrikstad: Mikjel Thorsrud (bortreist) Kontaktperson: Kent Ryne (69 60 87 37). Halden: Einar von Krogh (69608299) **Oppgaven er kontrollert:** Ja

Hjelpemidler:

- Godkjent kalkulator
- Ett A4-ark med valgfritt innhold (maskin eller håndskrevet, kan skrive på begge sider)
- Enten Tor Andersen: "Aktiv formelsamling i matematikk" eller "Gyldendals formelsamling i matematikk"

Om eksamensoppgaven:

Oppgavesettet består av 11 deloppgaver som i utgangspunktet vektes likt:
1, 2a, 2b, 3a, 3b, 3c, 4a, 4b, 5a, 5b, 5c.
Formelsamling (7 sider) er vedlagt.

Kandidaten må selv kontrollere at oppgavesettet er fullstendig

Oppgave 1

i) Les av store og lille halvakse til ellipsen i figuren til venstre. Skriv ned ligningen for ellipsen.

ii) Figuren til høyre viser et rektangel innskrevet i den samme ellipsen. Bruk Lagranges muliplikatormetode til å finne maksimumsarealet til et rektangel innskrevet i ellipsen. Tips: la hjørnet i første kvadrant ha koordinater (x, y), se figur. Rektangelets areal og betingelsen kan da uttrykkes ved x og y.
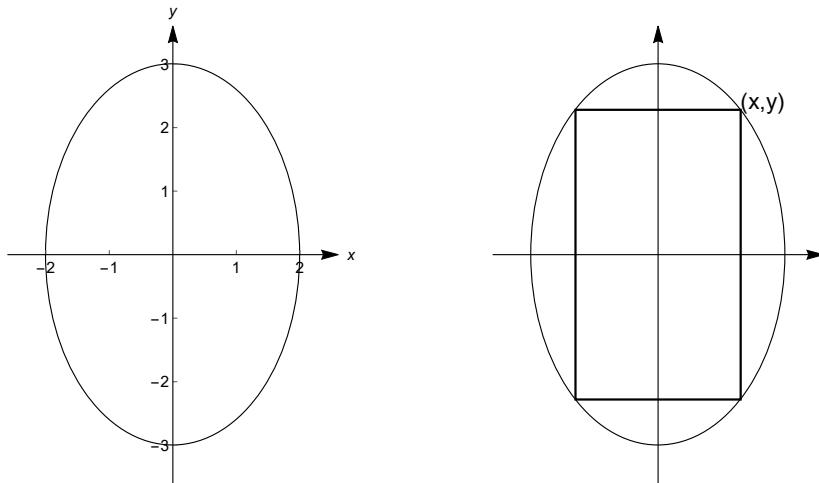
Oppgave 2

a) Regn ut dobbeltintegralet

$$\iint_R f(x, y)\, dA$$

hvor $f(x, y) = 3xy$ og R er er trekanten med hjørner i punktene $(0, 0)$, $(4, 0)$ og $(2, 2)$. Hva er middelverdien til funksjonen f over R?

b) Regn ut trippelintegralet

$$\int_0^1 \int_0^{\sqrt{x}} \int_0^y 2e^x\, dz\, dy\, dx$$

Vis alle mellomregninger.

Oppgave 3   En kurve C er gitt ved følgende parametrisering:

$$\mathbf{r} = 3\sin t\,\mathbf{i} + 5\cos t\,\mathbf{j} + 4\sin t\,\mathbf{k}\,, \quad 0 \leq t \leq \frac{\pi}{2}$$

a) Regn ut linjeintegralet

$$\int_C xyz\,ds$$

b) Regn ut virvlingen til det konservative kraftfeltet

$$\mathbf{F} = 2x\,\mathbf{i} + 2yze^{y^2}\,\mathbf{j} + e^{y^2}\,\mathbf{k}$$

Begrunn at feltet konservativt.

c) En partikkel som beveger seg langs kurven C er under påvirkning av kraftfeltet $\mathbf{F}$. Bestem arbeidet

$$\int_C \mathbf{F}\cdot d\mathbf{r}$$

Oppgave 4   Et vektorfelt er gitt ved

$$\mathbf{F} = (2x + 3x^2y^2 + e^{z\sin y})\,\mathbf{i} + (x\sin z - 2xy^3)\,\mathbf{j} + z\,\mathbf{k}$$

Flaten S er den delen av paraboloiden

$$z = x^2 + y^2$$

hvor $z \leq 4$. D er regionen i rommet avgrenset nedad av S og oppad av planet $z = 4$.

a) Regn ut divergensen til $\mathbf{F}$. Tegn en skisse og beskriv regionen D i passende koordinater. Bruk dette til å regne ut trippelintegralet

$$\iiint_D \nabla\cdot\mathbf{F}\,dV$$

b) Bruk divergensteoremet til å bestemme fluksen av $\mathbf{F}$ gjennom flaten S:

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma$$

Enhetsnormalvektoren peker vekk fra z-aksen, altså $\mathbf{n}\cdot(x\,\mathbf{i} + y\,\mathbf{j}) \geq 0$.
Hint: merk at S bare er en del av randflaten til D.

Oppgave 5 En kloss med masse m er koblet til en fjær. Bevegelsen er rettlinjet og klossens posisjon ved tidspunktet t er $x(t)$. Kreftene som virker på klossen er fjærkraften $F_k = -kx$ og luftmotstanden $F_b = -b\frac{dx}{dt}$, hvor k og b er to konstanter.

a)   i) Bruk Newton's andre lov til å skrive ned en andreordens bevegelsesligning for kloss-fjær systemet.

     ii) Skriv bevegelsesligningene om på dimensjonsløs form ved å innføre dimensjonsløse variable $\tilde{t}$ og $\tilde{x}$ definert

$$t = \tau \cdot \tilde{t}, \quad x = L \cdot \tilde{x},$$

hvor τ er en tidsskala med SI-enhet s (sekund) og L er en lengdeskala med SI-enhet m (meter). Vis at ligningen kan skrives på formen

$$\frac{d^2\tilde{x}}{d\tilde{t}^2} + \alpha\frac{d\tilde{x}}{d\tilde{t}} + \tilde{x} = 0$$

ved å velge en passende tidsskala τ. Skriv ned et uttrykk for α.

I oppgave b) og c) antar vi at $m = 1$ kg, $k = \frac{1}{4}$ N/m og $b = 1$ Ns/m.

b)   i) Bestem verdien til konstanten α og vis at den er dimensjonsløs. Vis at bevegelsesligningen i oppg. a) ii) kan skrives om til to koblede førsteordens differensialligninger for variablene $\tilde{x}$ og $\tilde{v} = \frac{d\tilde{x}}{d\tilde{t}}$:

$$\textbf{I.} \quad \frac{d\tilde{x}}{d\tilde{t}} = \tilde{v},$$

$$\textbf{II.} \quad \frac{d\tilde{v}}{d\tilde{t}} = -\tilde{x} - 2\tilde{v}$$

     ii) La $\tilde{x}_n$ og $\tilde{v}_n$ være numeriske tilnærminger til $\tilde{x}(\tilde{t}_n)$ og $\tilde{v}(\tilde{t}_n)$, hvor $\tilde{t}_n = n \cdot \Delta\tilde{t}$. Bruk Eulers metode til å uttrykke $\tilde{x}_1$ og $\tilde{v}_1$ ved initialbetingelsene $\tilde{x}_0$ og $\tilde{v}_0$. Ved å sette $\tilde{x}_0 = 0$ og $\tilde{v}_0 = 1$ i resultatet skal du få følgende uttrykk:

$$\tilde{x}_1 = \Delta\tilde{t},$$
$$\tilde{v}_1 = 1 - 2\Delta\tilde{t}$$

c) Med de antatte verdiene for m, k og b har differensialligningen følgende eksakte løsning:

$$\tilde{x}(\tilde{t}) = Ae^{-\tilde{t}} + B\tilde{t}e^{-\tilde{t}}$$

Anta initialbetingelser $\tilde{x}_0 = 0$ og $\tilde{v}_0 = 1$ og sammenlign den eksakte løsningen i tidspunktet $\tilde{t} = \Delta\tilde{t}$ med uttrykket for $\tilde{x}_1$ og $\tilde{v}_1$ i oppg. b) ii). Stemmer resultatene dine med at Eulers metode er en første-ordens numerisk metode?
Vi minner om Taylor-rekken til e^x i punktet $x = 0$:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \dots$$

Collection of formulas − Matematikk 3 (IRF30017)

Conic sections

Conic sections on standard form with foci on the x-axis:

$$\text{Ellipse:} \qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad a > b, \quad \text{foci:} \quad (\pm c, 0), \quad c = \sqrt{a^2 - b^2}.$$

$$\text{Hyperbola:} \qquad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \quad \text{foci:} \quad (\pm c, 0), \quad c = \sqrt{a^2 + b^2}, \quad \text{asymptotes:} \quad y = \pm(b/a)x.$$

$$\text{Parabola:} \qquad y = \frac{x^2}{4p}, \quad \text{focus:} \quad (0, p), \quad \text{directrix:} \quad y = -p.$$

In the case of the ellipse, a is called the *semimajor axis* and b the *semiminor axis*.

English − norwegian: conic section − kjeglesnitt, directrix − styrelinje, semimajor axis − store halvakse, semiminor axis − lille halvakse.

The method of Lagrange multipliers

Assume that $f(x_1, \ldots, x_n)$ and $g(x_1, \ldots, x_n)$ are differentiable functions and that $\nabla g \neq 0$ when $g = 0$. The stationary points of f subject to the constraint $g = 0$ are found by solving the $n + 1$ scalar equations

$$\nabla f = \lambda \nabla g, \quad g = 0$$

for the $n + 1$ unknowns $\lambda, x_1, \ldots, x_n$. The stationary points are *candidates* for local maxima and minima of f subject to $g = 0$.

Double and triple integrals

Cartesian (x, y, z), cylindrical (r, θ, z) and spherical (ρ, ϕ, θ) coordinates of a point P:

$$\begin{aligned}
\text{From cylindrical to Cartesian:} \qquad & x = r\cos\theta, \ y = r\sin\theta, \ z = z. \\
\text{From spherical to cylindrical:} \qquad & r = \rho\sin\phi, \ \theta = \theta, \ z = \rho\cos\phi. \\
\text{From spherical to Cartesian:} \qquad & x = \rho\sin\phi\cos\theta, \ y = \rho\sin\phi\sin\theta, \ z = \rho\cos\phi.
\end{aligned}$$

$r = \sqrt{x^2 + y^2}$ is the distance to the z axis and $\rho = \sqrt{x^2 + y^2 + z^2}$ is the distance to the origin ($|\overrightarrow{OP}|$).
$\theta \in [0, 2\pi]$ is the polar angular coordinate of the projection of P on the xy-plane and $\phi \in [0, \pi]$ is the angle between the z-axis and $\overrightarrow{OP}$.

Area and volume elements:

$$\begin{aligned}
dA &= dx\, dy = r\, dr\, d\theta = |J(u, v)|\, du\, dv, \\
dV &= dx\, dy\, dz = r\, dz\, dr\, d\theta = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta = |J(u, v, w)|\, du\, dv\, dw, \\
J(u, v) &= \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}, \quad J(u, v, w) = \frac{\partial(x, y, z)}{\partial(u, v, w)}
\end{aligned}$$

Applications of double and triple integrals:

$$\text{Area of } R: \quad A = \iint_R dA, \qquad\qquad \text{Volume of } D: \qquad V = \iiint_D dV$$

$$\text{Average of } f \text{ over } R: \quad \overline{f} = \frac{1}{A} \iint_R f(x, y)\, dA, \quad \text{Average of } f \text{ over } D: \quad \overline{f} = \frac{1}{V} \iiint_D f(x, y, z)\, dV$$

Object with mass density $\delta(x, y, z)$ occupying a region D in space:

$$\text{Mass:} \quad M = \iiint_D \delta(x, y, z)\, dV, \qquad \text{Center of mass:} \quad \overline{x} = \frac{M_{yz}}{M}, \quad \overline{y} = \frac{M_{xz}}{M}, \quad \overline{z} = \frac{M_{xy}}{M},$$

$$M_{yz} = \iiint_D x\, \delta(x, y, z)\, dV, \quad M_{xz} = \iiint_D y\, \delta(x, y, z)\, dV, \quad M_{xy} = \iiint_D z\, \delta(x, y, z)\, dV$$

Parametric curves and line integrals

Below the following parametrization of a curve C in space is assumed:

$$C: \quad \mathbf{r}(t) = g(t)\mathbf{i} + h(t)\mathbf{j} + k(t)\mathbf{k}, \qquad a \le t \le b$$

Tangent vector: $\quad \mathbf{v}(t) = \dfrac{d\mathbf{r}}{dt} = g'(t)\mathbf{i} + h'(t)\mathbf{j} + k'(t)\mathbf{k}, \quad$ Unit tangent vector: $\quad \mathbf{T} = \dfrac{\mathbf{v}}{|\mathbf{v}|}, \quad |\mathbf{v}| = \sqrt{\mathbf{v} \cdot \mathbf{v}},$

Arc length: $\quad L = \displaystyle\int_a^b |\mathbf{v}| \, dt, \qquad\qquad$ Arc length parameter: $\quad s(t) = \displaystyle\int_a^t |\mathbf{v}(t')| dt'$

Relations between differentials:

$$d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k}, \quad d\mathbf{r} = \mathbf{T}\,ds, \quad ds = |\mathbf{v}|dt$$

Line integral of scalar $f(x, y, z)$ along C:

$$\int_C f(x, y, z)\, ds = \int_a^b f(\mathbf{r}(t))\, |\mathbf{v}(t)|\, dt, \quad f(\mathbf{r}(t)) = f(g(t), h(t), k(t))$$

Line integral of vector field $\mathbf{F}(x, y, z) = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + P(x, y, z)\mathbf{k}$ along C:

$$\overbrace{\int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C M\,dx + N\,dy + P\,dz}^{\text{definitions}} = \overbrace{\int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{v}\, dt}^{\text{how to calculate}}$$

The line integral of the x-component of $\mathbf{F}$ along C:

$$\int_C M(x, y, z)\, dx = \int_a^b M(\mathbf{r}(t)) \frac{dx}{dt}\, dt = \int_a^b M(g(t), h(t), k(t))\, g'(t)\, dt$$

English − norwegian: line integral − linjeintegral, unit tangent vektor − enhets-tangentvektor, arc length − buelengde.

Names on line integrals: work, flow, circulation and flux

Let $\mathbf{F}$ be a vector field in $\mathbb{R}^n$ and C a parametrized curve in the same space. The line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

is called the

- *work* done by $\mathbf{F}$ on an object moving along the curve C if $\mathbf{F}$ is a force field

- *flow* of $\mathbf{F}$ along C if $\mathbf{F}$ is a velocity field

- *circulation* of $\mathbf{F}$ along C if $\mathbf{F}$ is a velocity field and C is a closed curve
  (for a closed curve the line integral is often written $\oint_C$)

Flux integral in two dimensions: Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ be a vector field and C a simple closed curve in the plane ($\mathbb{R}^2$) with unit normal $\mathbf{n}$ oriented outwards. The following line integral is the flux of $\mathbf{F}$ across the curve C:

$$\text{flux} = \oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \oint_C M\, dy - N\, dx$$

Flux integral in three dimensions: see surface integrals below.

English − norwegian: work − arbeid, flow − strøm, circulation − sirkulasjon, flux − fluks.

del, divergence and curl

Del operator:

$$\mathbb{R}^3: \quad \nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}, \qquad \mathbb{R}^n: \quad \nabla = \sum_{i=1}^{n} \mathbf{e}_i \frac{\partial}{\partial x^i}$$

The following definitions assume that $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is a vector field in space ($\mathbb{R}^3$), but the divergence generalizes naturally to a space of arbitrary dimensions ($\mathbb{R}^n$):

$$\text{Divergence of } \mathbf{F}: \quad \text{div}\,\mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}, \qquad \text{Curl of } \mathbf{F}: \quad \text{curl}\,\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix}$$

Identities: $\nabla \times (\nabla f) = 0, \quad \nabla \cdot (\nabla \times \mathbf{F}) = 0$

English − norwegian: del − nabla, divergence − divergens, curl − virvling.

Conservative fields and path independence

The following statements are equivalent if $\mathbf{F}$ is a vector field in space whose components have continuous partial derivatives in a connected and simply connected domain D and C is a curve in the same domain:

1. $\mathbf{F}$ is conservative
   (this is another way to say that the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is path independent)

2. $\mathbf{F}$ is curl-free, $\nabla \times \mathbf{F} = \vec{\mathbf{0}}$
   (this provides a component test for conservative fields, in the plane write $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + 0\mathbf{k}$)

3. $\mathbf{F}$ is a gradient field: $\mathbf{F} = \nabla f$
   (the function $f(x, y, z)$ is called a *potential function* for $\mathbf{F}$)

4. $\int_A^B \mathbf{F} \cdot d\mathbf{r} = f(B) - f(A)$ for all curves C from A to B

5. $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ for all closed curves C

English − norwegian: conservative − konservativ, path independent − veiuavhengig, curl-free − virvelfri.

Green's theorem

Let R be a region in the plane bounded by the piecewise smooth, simple closed curve C and let $\mathbf{F} = M(x,y)\mathbf{i} + N(x,y)\mathbf{j} + 0\mathbf{k}$ be a vector field with components M and N that have continuous partial derivatives.

Circulation-curl form:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \nabla \times \mathbf{F} \cdot \mathbf{k}\, dA$$

or

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

Flux-divergence form:

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_R \nabla \cdot \mathbf{F}\, dA$$

or

$$\oint_C M\,dy - N\,dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy$$

English − norwegian: piecewise smooth − stykkevis glatt, simple curve − enkel kurve.

Surface integrals

Let S be a smooth surface in space ($\mathbb{R}^3$). The area element $d\sigma$ depends on the description of S:

1) $d\sigma = |\mathbf{r}_u \times \mathbf{r}_v|\, du\, dv$ if S is given **parametrically** as $\mathbf{r}(u,v) = f_1(u,v)\mathbf{i} + f_2(u,v)\mathbf{j} + f_3(u,v)\mathbf{k}$

2) $d\sigma = \frac{|\nabla G|}{|\nabla G \cdot \mathbf{k}|}\, dx\, dy$ if S is given **implicitly** by the equation $G(x,y,z) = 0$

3) $d\sigma = \sqrt{g_x^2 + g_y^2 + 1}\, dx\, dy$ if S is given **explicitly** as the the graph $z = g(x,y)$

Below the case 3) of an explicitly defined surface is assumed. Let R be the shaddow of S on the xy-plane. The area of S is:
$$A = \iint_S d\sigma = \iint_R \sqrt{g_x^2 + g_y^2 + 1}\, dx\, dy$$

The integral of a scalar $f(x,y,z)$ over S:

$$\iint_S f(x,y,z)d\sigma = \iint_R f(x,y,g(x,y))\sqrt{g_x^2 + g_y^2 + 1}\, dx\, dy$$

A surface has two unit normal fields:

$$\mathbf{n} = \pm\frac{\nabla G}{|\nabla G|} = \pm\frac{-g_x\mathbf{i} - g_y\mathbf{j} + \mathbf{k}}{\sqrt{g_x^2 + g_y^2 + 1}}$$

For a given choice of $\mathbf{n}$ the *flux* of $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ across S is:

$$\text{Flux} = \iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \text{sgn}(\mathbf{n} \cdot \mathbf{k}) \iint_R -Mg_x - Ng_y + P\, dx\, dy$$

English − norwegian: surface integral − flateintegral, unit normal − enhetsnormal.

Stoke's theorem and the divergence theorem

Let S be an oriented piecewise smoooth surface in space having a piecewise smooth boundrary curve C that is directed counterclockwise relative to the unit normal field $\mathbf{n}$ of S. Stokes theorem:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma$$

Let D be a region in space with a piecewise smooth boundrary surface S having an outward unit normal field $\mathbf{n}$. Divergence theorem:

$$\oiint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iiint_D \nabla \cdot \mathbf{F}\, dV$$

In both thorems the components of $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ have continuous partial derivatives.

English − norwegian: boundrary − rand, boundrary curve − randkurve, boundrary surface − randflate, outward unit normal − utoverrettet enhetsnormal.

Modeling in physics

Numerical methods

Consider the first-order differential equation:

$$\frac{du}{dt} = f(u, t)$$

Let u_n be a numerical approximation to $u(t_n)$, where $t_n = t_0 + n\Delta t$.

Euler method: Use the tangent at the previous point to estimate the next:

$$u_{n+1} = u_n + \left(\frac{du}{dt}\right)_n \Delta t = u_n + f(u_n, t_n)\Delta t$$

Or more compactly:

$$u_{n+1} = u_n + f_n \Delta t$$

First order method (local error: $\sim \Delta t^2$, global error: $\sim \Delta t$).

Midpoint method: Use Euler's method with a half time step to estimate the slope at the midpoint (trial step), then apply this to estimate the next point:

$$u_{n+\frac{1}{2}} = u_n + \left(\frac{du}{dt}\right)_n \frac{\Delta t}{2} = u_n + \frac{1}{2}f(u_n, t_n)\Delta t, \quad \text{(trial step)},$$

$$u_{n+1} = u_n + \left(\frac{du}{dt}\right)_{n+\frac{1}{2}} \Delta t = u_n + f(u_{n+\frac{1}{2}}, t_n + \frac{\Delta t}{2})\Delta t$$

Or more compactly:

$$u_{n+\frac{1}{2}} = u_n + \frac{1}{2}f_n \Delta t, \quad \text{(trial step)},$$

$$u_{n+1} = u_n + f_{n+\frac{1}{2}} \Delta t$$

Second order method (local error: $\sim \Delta t^3$, global error: $\sim \Delta t^2$).

Higher order differential equations

A second order differential equation can be rewritten as a system of two coupled first order equations:

$$\frac{d^2 u}{dt^2} = f\left(u, \frac{du}{dt}, t\right) \quad \Longleftrightarrow \quad \textbf{I.} \quad \frac{du}{dt} = v,$$

$$\textbf{II.} \quad \frac{dv}{dt} = f(u, v, t)$$

The numerical schemes above can then be applied to find u_{n+1} and v_{n+1} from u_n and v_n.

Dimensionless variables

An ordinary differential equation for $x(t)$ can be written on dimensionless form by introducing a length scale L and time scale τ:

$$x = L\tilde{x}, \quad t = \tau\tilde{t} \quad \longrightarrow \quad \frac{dx}{dt} = \frac{L}{\tau}\frac{d\tilde{x}}{d\tilde{t}}, \quad \frac{d^2 x}{dt^2} = \frac{L}{\tau^2}\frac{d^2\tilde{x}}{d\tilde{t}^2},$$

where in SI units $\text{Dim}(x) = \text{Dim}(L) = \text{m}$, $\text{Dim}(t) = \text{Dim}(\tau) = \text{s}$ and $\text{Dim}(\tilde{x}) = \text{Dim}(\tilde{t}) = 1$.
SI base units: m, s, kg. SI derived units: N=kg·ms^{-2} (Newton's 2nd law), J=N·m (work-energy theorem).

Some solutions of selected differential equations

Harmonic oscillator equation   (ordinary, linear, homogeneous):

$$\frac{d^2x}{dt^2} + w^2 x = 0 \quad \rightarrow \quad x(t) = A\cos\left(wt + \phi\right)$$

Amplitude: A [m], angular frequency: w [rad/s], frequency: $f = \frac{w}{2\pi}$ [Hz], period: $T = \frac{1}{f} = \frac{2\pi}{w}$, phase: ϕ [rad].

One-dimensional wave equation   (partial, linear, homogeneous):

$$\frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2}$$

Mechanical waves on a string:

- Harmonic wave travelling to the right: $y(x,t) = A\cos\left(kx - wt + \phi\right), \quad w = v \cdot k.$
  Wave number: k [m^{-1}], wave length: $\lambda = \frac{2\pi}{k}$ [m].

- Standing waves with boundrary conditions $y(0,t) = y(L,t) = 0$:
  $y(x,t) = A\sin(kx) \cdot \cos(wt), \quad w = v \cdot k, \quad k = \frac{n\pi}{L}, \quad n = 1, 2, 3, \dots$

One-dimensional heat equation / diffusion equation   (partial, linear, homogeneous):

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \rightarrow \quad u(x,t) = A\sin(kx) \cdot e^{-(ck)^2 t}, \quad k = \frac{n\pi}{L}, \quad n = 1, 2, 3, \dots$$

The solutions above satisfy the boundrary conditions $u(0,t) = u(L,t) = 0$.

From previous courses

Scalar product and vector product

When $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$:

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| \cdot |\mathbf{b}| \cos\alpha = a_1b_1 + a_2b_2 + a_3b_3, \qquad \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}, \quad |\mathbf{a} \times \mathbf{b}| = |\mathbf{a}| \cdot |\mathbf{b}| \sin\alpha$$

Straight line in space

Parametrization of a line through the point $P_0(x_0, y_0, z_0)$ parallell to $\vec{v} = [a, b, c]$:

$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v} = (x_0 + at)\mathbf{i} + (y_0 + bt)\mathbf{j} + (z_0 + ct)\mathbf{k}, \quad -\infty \leq t \leq \infty$$

A possible parametrization of a straight line from $\mathbf{r}_1$ to $\mathbf{r}_2$:

$$\mathbf{r}(t) = \mathbf{r}_1 + (\mathbf{r}_2 - \mathbf{r}_1)t, \quad 0 \leq t \leq 1$$

Plane in space

Equation for a plane through the point $P_0(x_0, y_0, z_0)$ normal to $\vec{n} = [a, b, c]$:

$$\overrightarrow{P_0P} \cdot \vec{n} = 0 \quad \rightarrow \quad (x - x_0)a + (y - y_0)b + (z - z_0)c = 0$$

Circle in the plane

Equation for a circle with radius a and center in (x_0, y_0): $(x - x_0)^2 + (y - y_0)^2 = a^2$

Taylor expansion

Taylor series of a function $f(x)$ about the point $x = a$:

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x - a)^k = f(a) + f'(a)(x - a) + \frac{1}{2!}f''(a)(x - a)^2 + \ldots$$

Taylor polynom of degree n:

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x - a)^k = f(a) + f'(a)(x - a) + \frac{1}{2!}f''(a)(x - a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n$$

Linear approximation to $f(x)$ around $x = a$:

$$f(x) \simeq f(a) + f'(a)(x - a) \quad \text{if} \quad \left| \frac{1}{2}f''(a)(x - a)^2 \right| \ll |f'(a)(x - a)|$$

Some trigonometric identities

$$\sin^2 u + \cos^2 u = 1, \quad \sin(u + v) = \sin u \cos v + \cos u \sin v, \quad \cos(u + v) = \cos u \cos v - \sin u \sin v,$$
$$\sin(2u) = 2\sin u \cos u, \quad \cos(2u) = \cos^2 u - \sin^2 u, \quad \cos^2 u = (1 + \cos(2u))/2, \quad \sin^2 u = (1 - \cos(2u))/2$$