Høgskolen i Østfold

EKSAMEN

Emnekode:	Emnenavn:
IRF30017	Matematikk 3
Dato: 24.04.2019 **Sensurfrist:** 15.05.2019	**Eksamenstid:** 0900-1300
Antall oppgavesider: 3 **Antall vedleggsider:** 7	**Faglærer:** Fredrikstad: Mikjel Thorsrud (41 51 86 10) Halden: Einar von Krogh (69 60 82 99) **Oppgaven er kontrollert:** Ja

Hjelpemidler:

- Godkjent kalkulator
- Ett A4-ark med valgfritt innhold (maskin eller håndskrevet, kan skrive på begge sider)
- Enten Tor Andersen: "Aktiv formelsamling i matematikk" eller "Gyldendals formelsamling i matematikk"

Om eksamensoppgaven:

Oppgavesettet består av 11 deloppgaver som i utgangspunktet vektes likt:
1a, 1b, 1c, 2a, 2b, 2c, 2d, 2e, 3a, 3b, 3c.
Formelsamling (7 sider) er vedlagt.

Kandidaten må selv kontrollere at oppgavesettet er fullstendig

Oppgave 1

a) Regn ut trippelintegralet

$$\int_0^{2\pi} \int_0^{\pi/2} \int_0^3 \rho^2 \sin\phi \; d\rho\, d\phi\, d\theta$$

Vis alle mellomregninger. Integralet gir volumet til et objekt i rommet. Hva slags objekt?

b) R er området i planet avgrenset av linjene $y = x$, $y = 6 - 2x$ og y-aksen. Tegn en skisse av R og regn ut dobbeltintegralet

$$\iint_R 3xy\, dA$$

c) D er området i første oktant (der $x \geq 0,\ y \geq 0,\ z \geq 0$) som er avgrenset av planet $x + 2y + z = 4$ og koordinatplanene.

i) Tegn en skisse av området D. Bestem volumet til D på valgfri måte (det er ok å bruke kjente volumformler).

ii) Et legeme som okkuperer området D har innvendig temperatur gitt ved funksjonen $T(x, y, z) = 20 + z$. Skriv ned et beregningsklart trippelintegral som gir gjennomsnittstemperaturen $\overline{T}$ til legemet. Du trenger ikke regne ut integralet, men det skal kunne tastes direkte inn på en kalkulator og gi svaret $\overline{T} = 21$.

Oppgave 2

a) En ellipse er bestemt av ligningen

$$x^2 + 4y^2 = 16\,.$$

Skriv ligningen på standardform og bestem store og lille halvakse. Bestem koordinatene til brennpunktene.

b) En kurve C har følgende parametrisering:

$$\mathbf{r}(t) = 4\cos\pi t\;\mathbf{i} + 2\sin\pi t\;\mathbf{j}\,, \quad 0 \le t \le 2\,.$$

i) Vis ved innsetting i ligningen for ellipsen at kurven C er ellipsen i oppg. a).

ii) La A være punktet på ellipsen som tilsvarer parameterverdien $t = 1/6$. Skriv ned de kartesiske koordinatene (x, y) til punktet A.

c) i) La $\mathbf{v}(t) = \frac{d\mathbf{r}}{dt}$ være tangentvektor til kurven C definert over. Regn ut enhets-tangentvektor $\mathbf{T}(t) = \frac{\mathbf{v}}{|\mathbf{v}|}$ i punktet A, altså $\mathbf{T}(1/6)$.

ii) Tegn en skisse av kurven C (ellipsen). Marker punktet A og tegn inn enhets-tangentvektoren i dette punktet. Marker også brennpunktene til ellipsen.

d) Et vektorfelt i rommet er definert ved

$$\mathbf{F}(x, y, z) = -ye^{2z}\;\mathbf{i} + (x + yz)\;\mathbf{j} + z\;\mathbf{k}$$

Regn ut divergensen og virvlingen til $\mathbf{F}$. Er vektorfeltet konservativt?

e) Flaten S er øvre halvdel $(z \ge 0)$ av ellipsoiden

$$\frac{x^2}{4^2} + \frac{y^2}{2^2} + \frac{z^2}{3^2} = 1$$

Bruk Stokes' teorem til å regne ut flateintegralet

$$\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\; d\sigma$$

hvor $\mathbf{F}$ er vektorfeltet definert over og $\mathbf{n}$ er enhets-normalvektor til S og peker "oppover" $(\mathbf{n} \cdot \mathbf{k} \ge 0)$.

Oppgave 3 En nyttårsrakett med masse m går tom for krutt mens den er på vei rett oppover. Kreftene som virker på raketten er da tyngdekraften og luftmotstand som begge virker i retning rett nedover. Tyngdekraften har størrelse mg, hvor g er tyngdeakselerasjonen. Vi antar en kvadratisk modell for luftmotstanden med størrelse bv^2, hvor $v = \frac{dy}{dt}$ er rakettens hastighet og b er en positiv koeffisient.

a) i) Skriv ned en andre-ordens differensialligning for posisjonen $y(t)$ ved å bruke Newtons andre lov.

ii) Vi innfører dimensjonsløse variable $\tilde{t}$ og $\tilde{y}$ definert

$$t = \tau \cdot \tilde{t}, \quad y = L \cdot \tilde{y},$$

hvor τ er en tidsskala med SI-enhet s (sekund) og L er en lengdeskala med SI-enhet m (meter). Vis at differensialligningen kan skrives på dimensjonsløs form som

$$\frac{d^2\tilde{y}}{d\tilde{t}^2} + \left(\frac{d\tilde{y}}{d\tilde{t}}\right)^2 + 1 = 0 \ .$$

Bestem τ og L uttrykt ved b, m og g.

b) Ved tidspunktet $\tilde{t} = 0$ er posisjonen $\tilde{y}_0 = 1$ og hastigheten $\tilde{v}_0 = 1$. Bruk midtpunktmetoden til å finne en tilnærmet verdi for posisjonen $\tilde{y}$ ved tidspunktet $\tilde{t} = 0.1$. Bruk kun ett tidssteg, dvs. regn ut $\tilde{y}_1$ ved å bruke tidssteget $\Delta\tilde{t} = 0.1$.
Tips: merk at vi i denne og den neste deloppgaven jobber utelukkende med de dimensjonsløse størrelsene. Ta derfor utgangspunkt i den dimensjonsløse differensialligningen skrevet ned i oppg. a). Begynn som vanlig med å skrive differensialligningen om til to koblede første-ordens ligninger for variablene $\tilde{y}$ og $\tilde{v}$.

c) Differensialligningen i oppg a) har følgende eksakte løsning:

$$\tilde{y}(\tilde{t}) = A + \ln[\cos(\tilde{t} - B)]$$

hvor A og B er konstanter. Bestem A og B ved å bruke initialbetingelsene i oppg. b). Regn ut feilen $|\tilde{y}(0.1) - \tilde{y}_1|$ til tilnærmingen i oppg. b). Stemmer feilens størrelse med at midtpunktmetoden er en andre-ordens numerisk metode?

Collection of formulas – Matematikk 3 (IRF30017)

Conic sections

Conic sections on standard form with foci on the x-axis:

Ellipse: $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1, \quad a>b, \quad$ foci: $(\pm c,0), \quad c=\sqrt{a^2-b^2}.$

Hyperbola: $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1, \quad$ foci: $(\pm c,0), \quad c=\sqrt{a^2+b^2}, \quad$ asymptotes : $\quad y=\pm(b/a)x.$

Parabola: $y=\dfrac{x^2}{4p}, \quad$ focus: $(0,p), \quad$ directrix: $y=-p.$

In the case of the ellipse, a is called the *semimajor axis* and b the *semiminor axis.*

English – norwegian: conic section – kjeglesnitt, directrix – styrelinje, semimajor axis – store halvakse, semiminor axis – lille halvakse.

The method of Lagrange multipliers

Assume that $f(x_1,\ldots,x_n)$ and $g(x_1,\ldots,x_n)$ are differentiable functions and that $\nabla g\neq 0$ when $g=0$. The stationary points of f subject to the constraint $g=0$ are found by solving the $n+1$ scalar equations

$$\nabla f=\lambda\nabla g, \quad g=0$$

for the $n+1$ unknowns $\lambda,x_1,\ldots,x_n$. The stationary points are *candidates* for local maxima and minima of f subject to $g=0$.

Double and triple integrals

Cartesian (x,y,z), cylindrical (r,θ,z) and spherical (ρ,ϕ,θ) coordinates of a point P:

From cylindrical to Cartesian: $x=r\cos\theta,\ y=r\sin\theta,\ z=z.$

From spherical to cylindrical: $r=\rho\sin\phi,\ \theta=\theta,\ z=\rho\cos\phi.$

From spherical to Cartesian: $x=\rho\sin\phi\cos\theta,\ y=\rho\sin\phi\sin\theta,\ z=\rho\cos\phi.$

$r=\sqrt{x^2+y^2}$ is the distance to the z axis and $\rho=\sqrt{x^2+y^2+z^2}$ is the distance to the origin ($|\overrightarrow{OP}|$). $\theta\in[0,2\pi]$ is the polar angular coordinate of the projection of P on the xy-plane and $\phi\in[0,\pi]$ is the angle between the z-axis and $\overrightarrow{OP}$.

Area and volume elements:

$$dA=dx\,dy=r\;dr\,d\theta=|J(u,v)|\;du\,dv,$$
$$dV=dx\,dy\,dz=r\;dz\,dr\,d\theta=\rho^2\sin\phi\;d\rho\,d\phi\,d\theta=|J(u,v,w)|\;du\,dv\,dw,$$
$$J(u,v)=\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}, \quad J(u,v,w)=\frac{\partial(x,y,z)}{\partial(u,v,w)}$$

Applications of double and triple integrals:

Area of R: $\quad A=\displaystyle\iint_R dA,$ $\qquad$ Volume of D: $\quad V=\displaystyle\iiint_D dV$

Average of f over R: $\quad \overline{f}=\dfrac{1}{A}\displaystyle\iint_R f(x,y)\,dA,$ $\qquad$ Average of f over D: $\quad \overline{f}=\dfrac{1}{V}\displaystyle\iiint_D f(x,y,z)\,dV$

Object with mass density $\delta(x,y,z)$ occupying a region D in space:

Mass: $\quad M=\displaystyle\iiint_D \delta(x,y,z)\,dV,$ $\qquad$ Center of mass: $\quad \overline{x}=\dfrac{M_{yz}}{M}, \quad \overline{y}=\dfrac{M_{xz}}{M}, \quad \overline{z}=\dfrac{M_{xy}}{M},$

$$M_{yz}=\iiint_D x\,\delta(x,y,z)\,dV, \quad M_{xz}=\iiint_D y\,\delta(x,y,z)\,dV, \quad M_{xy}=\iiint_D z\,\delta(x,y,z)\,dV$$

Parametric curves and line integrals

Below the following parametrization of a curve C in space is assumed:

$$C: \quad \mathbf{r}(t) = g(t)\mathbf{i} + h(t)\mathbf{j} + k(t)\mathbf{k}, \qquad a \le t \le b$$

Tangent vector: $\mathbf{v}(t) = \dfrac{d\mathbf{r}}{dt} = g'(t)\mathbf{i} + h'(t)\mathbf{j} + k'(t)\mathbf{k}$, Unit tangent vector: $\mathbf{T} = \dfrac{\mathbf{v}}{|\mathbf{v}|}$, $|\mathbf{v}| = \sqrt{\mathbf{v}\cdot\mathbf{v}}$,

Arc length: $L = \displaystyle\int_a^b |\mathbf{v}|\, dt$, Arc length parameter: $s(t) = \displaystyle\int_a^t |\mathbf{v}(t')|dt'$

Relations between differentials:

$$d\mathbf{r} = dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}, \quad d\mathbf{r} = \mathbf{T}ds, \quad ds = |\mathbf{v}|dt$$

Line integral of scalar $f(x,y,z)$ along C:

$$\int_C f(x,y,z)\, ds = \int_a^b f(\mathbf{r}(t))\, |\mathbf{v}(t)|\, dt, \quad f(\mathbf{r}(t)) = f(g(t), h(t), k(t))$$

Line integral of vector field $\mathbf{F}(x,y,z) = M(x,y,z)\mathbf{i} + N(x,y,z)\mathbf{j} + P(x,y,z)\mathbf{k}$ along C:

$$\overbrace{\int_C \mathbf{F}\cdot\mathbf{T}\, ds = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_C M dx + N dy + P dz}^{\text{definitions}} = \overbrace{\int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{v}\, dt}^{\text{how to calculate}}$$

The line integral of the x-component of $\mathbf{F}$ along C:

$$\int_C M(x,y,z)\, dx = \int_a^b M(\mathbf{r}(t))\frac{dx}{dt}\, dt = \int_a^b M(g(t), h(t), k(t))\, g'(t)\, dt$$

English – norwegian: line integral – linjeintegral, unit tangent vektor – enhets-tangentvektor, arc length – buelengde.

Names on line integrals: work, flow, circulation and flux

Let $\mathbf{F}$ be a vector field in $\mathbb{R}^n$ and C a parametrized curve in the same space. The line integral

$$\int_C \mathbf{F}\cdot d\mathbf{r}$$

is called the

- *work* done by $\mathbf{F}$ on an object moving along the curve C if $\mathbf{F}$ is a force field
- *flow* of $\mathbf{F}$ along C if $\mathbf{F}$ is a velocity field
- *circulation* of $\mathbf{F}$ along C if $\mathbf{F}$ is a velocity field and C is a closed curve (for a closed curve the line integral is often written $\oint_C$)

Flux integral in two dimensions: Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ be a vector field and C a simple closed curve in the plane ($\mathbb{R}^2$) with unit normal $\mathbf{n}$ oriented outwards. The following line integral is the flux of $\mathbf{F}$ across the curve C:

$$\text{flux} = \oint_C \mathbf{F}\cdot\mathbf{n}\, ds = \oint_C M\, dy - N\, dx$$

Flux integral in three dimensions: see surface integrals below.

English – norwegian: work – arbeid, flow – strøm, circulation – sirkulasjon, flux – fluks.

del, divergence and curl

Del operator:

$$\mathbb{R}^3: \quad \nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}, \qquad \mathbb{R}^n: \quad \nabla = \sum_{i=1}^{n} \mathbf{e}_i \frac{\partial}{\partial x^i}$$

The following definitions assume that $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is a vector field in space ($\mathbb{R}^3$), but the divergence generalizes naturally to a space of arbitrary dimensions ($\mathbb{R}^n$):

$$\text{Divergence of } \mathbf{F}: \quad \operatorname{div}\mathbf{F} = \nabla\cdot\mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}, \qquad \text{Curl of } \mathbf{F}: \quad \operatorname{curl}\mathbf{F} = \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix}$$

Identities: $\nabla\times(\nabla f) = 0, \quad \nabla\cdot(\nabla\times\mathbf{F}) = 0$
English – norwegian: del – nabla, divergence – divergens, curl – virvling.

Conservative fields and path independence

The following statements are equivalent if $\mathbf{F}$ is a vector field in space whose components have continuous partial derivatives in a connected and simply connected domain D and C is a curve in the same domain:

1. $\mathbf{F}$ is conservative
   (this is another way to say that the integral $\int_C \mathbf{F}\cdot d\mathbf{r}$ is path independent)
2. $\mathbf{F}$ is curl-free, $\nabla\times\mathbf{F} = \vec{\mathbf{0}}$
   (this provides a component test for conservative fields, in the plane write $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + 0\mathbf{k}$)
3. $\mathbf{F}$ is a gradient field: $\mathbf{F} = \nabla f$
   (the function $f(x, y, z)$ is called a *potential function* for $\mathbf{F}$)
4. $\int_A^B \mathbf{F}\cdot d\mathbf{r} = f(B) - f(A)$ for all curves C from A to B
5. $\oint_C \mathbf{F}\cdot d\mathbf{r} = 0$ for all closed curves C

English – norwegian: conservative – konservativ, path independent – veiuavhengig, curl-free – virvelfri.

Green's theorem

Let R be a region in the plane bounded by the piecewise smooth, simple closed curve C and let $\mathbf{F} = M(x, y)\mathbf{i} + N(x, y)\mathbf{j} + 0\mathbf{k}$ be a vector field with components M and N that have continuous partial derivatives.

Circulation-curl form:

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_R \nabla\times\mathbf{F}\cdot\mathbf{k}\, dA$$

or

$$\oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$$

Flux-divergence form:

$$\oint_C \mathbf{F}\cdot\mathbf{n}\, ds = \iint_R \nabla\cdot\mathbf{F}\, dA$$

or

$$\oint_C M dy - N dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\, dy$$

English – norwegian: piecewise smooth – stykkevis glatt, simple curve – enkel kurve.

Surface integrals

Let S be a smooth surface in space ($\mathbb{R}^3$). The area element $d\sigma$ depends on the description of S:

1) $d\sigma = |\mathbf{r}_u \times \mathbf{r}_v|\, du\, dv$ if S is given **parametrically** as $\mathbf{r}(u,v) = f_1(u,v)\mathbf{i} + f_2(u,v)\mathbf{j} + f_3(u,v)\mathbf{k}$

2) $d\sigma = \frac{|\nabla G|}{|\nabla G \cdot \mathbf{k}|}\, dx\, dy$ if S is given **implicitly** by the equation $G(x,y,z) = 0$

3) $d\sigma = \sqrt{g_x^2 + g_y^2 + 1}\, dx\, dy$ if S is given **explicitly** as the the graph $z = g(x,y)$

Below the case 3) of an explicitly defined surface is assumed. Let R be the shaddow of S on the xy-plane. The area of S is:

$$A = \iint_S d\sigma = \iint_R \sqrt{g_x^2 + g_y^2 + 1}\, dx\, dy$$

The integral of a scalar $f(x,y,z)$ over S:

$$\iint_S f(x,y,z) d\sigma = \iint_R f(x,y,g(x,y)) \sqrt{g_x^2 + g_y^2 + 1}\, dx\, dy$$

A surface has two unit normal fields:

$$\mathbf{n} = \pm \frac{\nabla G}{|\nabla G|} = \pm \frac{-g_x \mathbf{i} - g_y \mathbf{j} + \mathbf{k}}{\sqrt{g_x^2 + g_y^2 + 1}}$$

For a given choice of $\mathbf{n}$ the *flux* of $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ across S is:

$$\text{Flux} = \iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \text{sgn}(\mathbf{n} \cdot \mathbf{k}) \iint_R -M g_x - N g_y + P\, dx\, dy$$

English – norwegian: surface integral – flateintegral, unit normal – enhetsnormal.

Stoke's theorem and the divergence theorem

Let S be an oriented piecewise smoooth surface in space having a piecewise smooth boundrary curve C that is directed counterclockwise relative to the unit normal field $\mathbf{n}$ of S. Stokes theorem:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma$$

Let D be a region in space with a piecewise smooth boundrary surface S having an outward unit normal field $\mathbf{n}$. Divergence theorem:

$$\oiint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iiint_D \nabla \cdot \mathbf{F}\, dV$$

In both thorems the components of $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ have continuous partial derivatives.

English – norwegian: boundrary – rand, boundrary curve – randkurve, boundrary surface – randflate, outward unit normal – utoverrettet enhetsnormal.

Modeling in physics

Numerical methods

Consider the first-order differential equation:

$$\frac{du}{dt} = f(u,t)$$

Let u_n be a numerical approximation to $u(t_n)$, where $t_n = t_0 + n\Delta t$.

Euler method: Use the tangent at the previous point to estimate the next:

$$u_{n+1} = u_n + \left(\frac{du}{dt}\right)_n \Delta t = u_n + f(u_n, t_n)\Delta t$$

Or more compactly:

$$u_{n+1} = u_n + f_n \Delta t$$

First order method (local error: $\sim \Delta t^2$, global error: $\sim \Delta t$).

Midpoint method: Use Euler's method with a half time step to estimate the slope at the midpoint (trial step), then apply this to estimate the next point:

$$\begin{aligned} u_{n+\frac{1}{2}} &= u_n + \left(\frac{du}{dt}\right)_n \frac{\Delta t}{2} = u_n + \frac{1}{2} f(u_n, t_n)\Delta t, \quad \text{(trial step)}, \\ u_{n+1} &= u_n + \left(\frac{du}{dt}\right)_{n+\frac{1}{2}} \Delta t = u_n + f(u_{n+\frac{1}{2}}, t_n + \frac{\Delta t}{2})\Delta t \end{aligned}$$

Or more compactly:

$$\begin{aligned} u_{n+\frac{1}{2}} &= u_n + \frac{1}{2} f_n \Delta t, \quad \text{(trial step)}, \\ u_{n+1} &= u_n + f_{n+\frac{1}{2}} \Delta t \end{aligned}$$

Second order method (local error: $\sim \Delta t^3$, global error: $\sim \Delta t^2$).

Higher order differential equations

A second order differential equation can be rewritten as a system of two coupled first order equations:

$$\frac{d^2u}{dt^2} = f\left(u, \frac{du}{dt}, t\right) \quad \Longleftrightarrow \quad \textbf{I.} \quad \frac{du}{dt} = v, \qquad \textbf{II.} \quad \frac{dv}{dt} = f(u, v, t)$$

The numerical schemes above can then be applied to find u_{n+1} and v_{n+1} from u_n and v_n.

Dimensionless variables

An ordinary differential equation for $x(t)$ can be written on dimensionless form by introducing a length scale L and time scale τ:

$$x = L\tilde{x}, \quad t = \tau\tilde{t} \qquad \longrightarrow \qquad \frac{dx}{dt} = \frac{L}{\tau}\frac{d\tilde{x}}{d\tilde{t}}, \quad \frac{d^2x}{dt^2} = \frac{L}{\tau^2}\frac{d^2\tilde{x}}{d\tilde{t}^2},$$

where in SI units $\mathrm{Dim}(x) = \mathrm{Dim}(L) = \mathrm{m}$, $\mathrm{Dim}(t) = \mathrm{Dim}(\tau) = \mathrm{s}$ and $\mathrm{Dim}(\tilde{x}) = \mathrm{Dim}(\tilde{t}) = 1$.
SI base units: m, s, kg. SI derived units: N=kg·ms^{-2} (Newton's 2nd law), J=N·m (work-energy theorem).

Some solutions of selected differential equations

Harmonic oscillator equation (ordinary, linear, homogeneous):

$$\frac{d^2x}{dt^2} + w^2 x = 0 \quad \rightarrow \quad x(t) = A\cos(wt + \phi)$$

Amplitude: A [m], angular frequency: w [rad/s], frequency: $f = \frac{w}{2\pi}$ [Hz], period: $T = \frac{1}{f} = \frac{2\pi}{w}$, phase: ϕ [rad].

One-dimensional wave equation (partial, linear, homogeneous):

$$\frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2}$$

Mechanical waves on a string:

- Harmonic wave travelling to the right: $y(x,t) = A\cos(kx - wt + \phi)$, $w = v \cdot k$.
  Wave number: k [m^{-1}], wave length: $\lambda = \frac{2\pi}{k}$ [m].
- Standing waves with boundrary conditions $y(0,t) = y(L,t) = 0$:
  $y(x,t) = A\sin(kx) \cdot \cos(wt)$, $w = v \cdot k$, $k = \frac{n\pi}{L}$, $n = 1, 2, 3, \ldots$

One-dimensional heat equation / diffusion equation (partial, linear, homogeneous):

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \rightarrow \quad u(x,t) = A\sin(kx) \cdot e^{-(ck)^2 t}, \quad k = \frac{n\pi}{L}, \quad n = 1, 2, 3, \ldots$$

The solutions above satisfy the boundrary conditions $u(0,t) = u(L,t) = 0$.

From previous courses

Scalar product and vector product

When $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$:

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| \cdot |\mathbf{b}| \cos\alpha = a_1b_1 + a_2b_2 + a_3b_3, \qquad \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}, \quad |\mathbf{a} \times \mathbf{b}| = |\mathbf{a}| \cdot |\mathbf{b}| \sin\alpha$$

Straight line in space

Parametrization of a line through the point $P_0(x_0, y_0, z_0)$ parallell to $\vec{v} = [a, b, c]$:

$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v} = (x_0 + at)\mathbf{i} + (y_0 + bt)\mathbf{j} + (z_0 + ct)\mathbf{k}, \quad -\infty \leq t \leq \infty$$

A possible parametrization of a straight line from $\mathbf{r}_1$ to $\mathbf{r}_2$:

$$\mathbf{r}(t) = \mathbf{r}_1 + (\mathbf{r}_2 - \mathbf{r}_1)t, \quad 0 \leq t \leq 1$$

Plane in space

Equation for a plane through the point $P_0(x_0, y_0, z_0)$ normal to $\vec{n} = [a, b, c]$:

$$\overrightarrow{P_0P} \cdot \vec{n} = 0 \quad \rightarrow \quad (x - x_0)a + (y - y_0)b + (z - z_0)c = 0$$

Circle in the plane

Equation for a circle with radius a and center in (x_0, y_0): $(x - x_0)^2 + (y - y_0)^2 = a^2$

Taylor expansion

Taylor series of a function $f(x)$ about the point $x = a$:

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x - a)^k = f(a) + f'(a)(x - a) + \frac{1}{2!}f''(a)(x - a)^2 + \dots$$

Taylor polynom of degree n:

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x - a)^k = f(a) + f'(a)(x - a) + \frac{1}{2!}f''(a)(x - a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n$$

Linear approximation to $f(x)$ around $x = a$:

$$f(x) \simeq f(a) + f'(a)(x - a) \quad \text{if} \quad \left|\frac{1}{2}f''(a)(x - a)^2\right| \ll |f'(a)(x - a)|$$

Some trigonometric identities

$$\sin^2 u + \cos^2 u = 1, \quad \sin(u + v) = \sin u \, \cos v + \cos u \, \sin v, \quad \cos(u + v) = \cos u \, \cos v - \sin u \, \sin v,$$
$$\sin(2u) = 2 \sin u \cos u, \quad \cos(2u) = \cos^2 u - \sin^2 u, \quad \cos^2 u = (1 + \cos(2u))/2, \quad \sin^2 u = (1 - \cos(2u))/2$$