LF, Matematikk 1, 2019 V

1a)

$$z^3 = 8i$$

$$z_n^3 = 8\,e^{i\left(\frac{\pi}{2} + 2\pi n\right)}$$

$$z_n = 8^{\frac{1}{3}}\,e^{i\left(\frac{\pi}{2} + 2\pi n\right)\cdot\frac{1}{3}}$$

$$z_n = 2\cdot e^{i\left(\frac{\pi}{6} + \frac{2\pi}{3}n\right)}$$

$$z_0 = 2\cdot e^{i\frac{\pi}{6}} = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = 2\left(\frac{\sqrt{3}}{2} + i\cdot\frac{1}{2}\right)$$

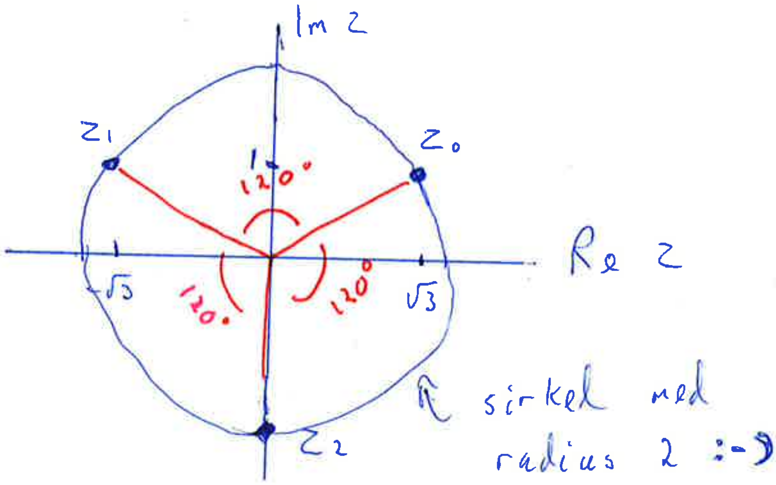
$$z_0 = \underline{\underline{\sqrt{3} + i}}$$

$$z_1 = 2\cdot e^{i\left(\frac{\pi}{6} + \frac{2\pi}{3}\right)} = 2\cdot e^{i\frac{5\pi}{6}} = 2\cdot\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$$

$$z_1 = 2\left(-\frac{\sqrt{3}}{2} + i\cdot\frac{1}{2}\right) = \underline{\underline{-\sqrt{3} + i}}$$

$$z_2 = 2\cdot e^{i\left(\frac{\pi}{6} + \frac{4\pi}{3}\right)} = 2\cdot e^{\frac{3\pi}{2}i} = 2\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right)$$

$$z_2 = 2\left(0 - i\right) = \underline{\underline{-2i}}$$



1 b) "$\frac{0}{0}$"-uttrykk

i) $\lim\limits_{x \to 0} \dfrac{\sin \frac{\pi x}{6}}{x} \overset{l'H}{=} \lim\limits_{x \to 0} \dfrac{\frac{\pi}{6} \cdot \cos \frac{\pi x}{6}}{1} = \dfrac{\pi}{6} \cdot \cos 0$

$= \underline{\underline{\dfrac{\pi}{6}}}$

ii) $\lim\limits_{x \to \infty} \dfrac{5x - x^3}{3x^3 + x^2 + 1}$ deler på ledende potens x^3

$= \lim\limits_{x \to \infty} \dfrac{\frac{5}{x^2} - 1}{3 + \frac{1}{x} + \frac{1}{x^3}} = \dfrac{0 - 1}{3 + 0 + 0} = \underline{\underline{-\dfrac{1}{3}}}$

1 c)

i) $f(x) = x^2 \cdot e^x$, $f(1) = e$

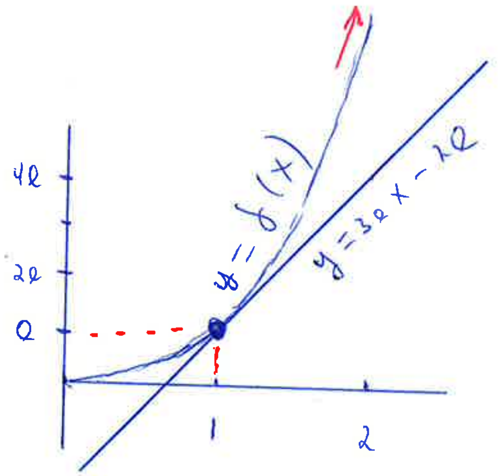
$$f'(x) = (x^2)' \cdot e^x + x^2 \cdot (e^x)'$$
$$= 2x\,e^x + x^2 e^x$$

$$f'(1) = 2e + e = 3e$$

Linear tilnærming i punktet $x = 1$:

$$f(x) \simeq f(1) + f'(1) \cdot (x-1)$$
$$= e + 3e(x-1)$$
$$= 3ex - 2e$$

ii)



Fikk ikke plass til å tegne grafen $y = f(x)$ helt til punktet $(2, 4e^2)$, men vi ser grafisk at gyldighetsområdet til tangent-tilnærm-ingen ikke går helt til $x = 2$!

1 d) i) $\int_0^1 \sin \pi x + \sqrt{2x}\, dx = \int_0^1 \sin \pi x + \sqrt{2} \cdot x^{\frac{1}{2}}\, dx$

$= \left[-\frac{1}{\pi} \cos \pi x + \sqrt{2} \cdot \frac{2}{3} x^{\frac{3}{2}} \right]_0^1$

$= \left(-\frac{1}{\pi} \cos \pi + \sqrt{2} \cdot \frac{2}{3} \cdot 1 \right) - \left(-\frac{1}{\pi} \cos 0 + 0 \right)$

$= \underline{\underline{\frac{2}{\pi} + \frac{2\sqrt{2}}{3}}}$

ii) $\int \frac{2x}{1+x^2}\, dx$

$\boxed{u = 1 + x^2 ,\quad \frac{du}{dx} = 2x ,\quad du = 2x\, dx}$

$= \int \frac{1}{1+x^2}\, 2x\, dx = \int \frac{1}{u}\, du$

$= \ln |u| + c$

$= \ln |1 + x^2| + c$ $\quad 1 + x^2 > 0$

$= \underline{\underline{\ln (1 + x^2) + c}}$

2 a) Simpson's regel med 4 delintervaller:

$$\int_0^4 e^{\sqrt{x}}\,dx \simeq \frac{\Delta x}{3}\left[y_0 + 4y_1 + 2y_2 + 4y_3 + y_4\right]$$

$$= \frac{1}{3}\left(e^0 + 4\cdot e^1 + 2e^{\sqrt{2}} + 4e^{\sqrt{3}} + e^2\right)$$

$$\simeq \underline{\underline{16{,}7}}$$

2 b) Variabelskifte:

$$u = \sqrt{x}\,, \quad \frac{du}{dx} = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2\sqrt{x}}$$

$$\Rightarrow dx = 2\sqrt{x}\,du = 2u\,du$$

Nye grenser: $u(0) = \sqrt{0} = 0$

$u(4) = \sqrt{4} = 2$

gir

$$\int_0^4 e^{\sqrt{x}}\,dx = \int_0^2 e^u\,2u\,du$$

delvis int

$$= \int_0^2 2u\cdot e^u\,du = \left[2u\cdot e^u\right]_0^2 - \int_0^2 2\cdot e^u\,du$$

$$= 2\cdot 2\cdot e^2 - 0 - \left[2e^u\right]_0^2 = 4e^2 - (2e^2 - 2)$$

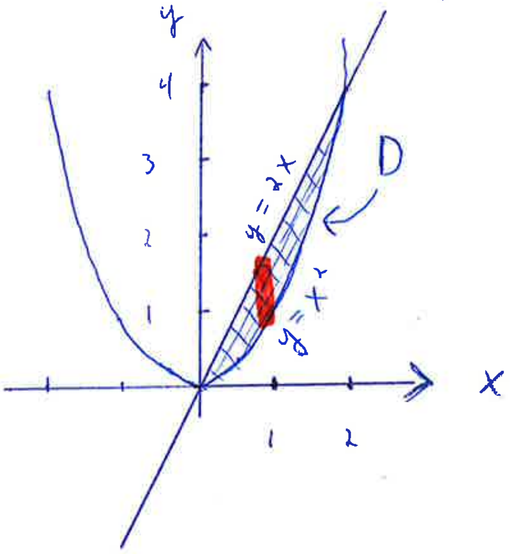
$$= \underline{\underline{2e^2 + 2}} \leftarrow \simeq 16{,}8$$

for å sammenligne med Simpson's regel

3 a) Skjæringspunkter:

$$x^2 = 2x$$

$$x(x-2) = 0 \Rightarrow x_1 = 0, \quad x_2 = 2$$

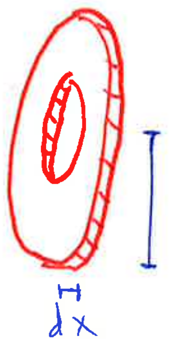


Arealet til D:

$$\int_0^2 2x - x^2 \, dx = \left[x^2 - \frac{1}{3}x^3\right]_0^2$$

$$= 4 - \frac{1}{3} \cdot 8 = \underline{\underline{\frac{4}{3}}}$$

3 b) Roteres den røde stolpen i skissen over rundt x-aksen fås en sylinderskive med hull i midten:



$$dV = \pi (2x)^2 \, dx - \pi (x^2)^2 \, dx$$

$$dV = \pi (4x^2 - x^4) \, dx$$

Volum til omdreiningslegeme:

$$V_x = \int dV = \int_0^2 \pi (4x^2 - x^4) \, dx = \pi \left[\frac{4}{3}x^3 - \frac{1}{5}x^5\right]_0^2$$

$$= \pi \left(\frac{4 \cdot 8}{3} - \frac{32}{5}\right) = \underline{\underline{\frac{64\pi}{15}}}$$

4 a) $y'' - 6y' + 9y = 0$

Karakteristisk ligning:

$$r^2 - 6r + 9 = 0$$

$$(r-3)^2 = 0$$

$$\Rightarrow r = 3 \quad (\text{en rot})$$

Generell løsning av diff. ligningen:

$$y(x) = A \cdot e^{3x} + Bx \cdot e^{3x}$$

4c) $27x^2$ er et 2. gr. polynom så antar at partikulær løsningen også er et 2. gr. polynom:

$$y_p = Cx^2 + Dx + E$$

$$y_p' = 2Cx + D$$

$$y_p'' = 2C$$

Innsatt:

$$2C - 6[2Cx + D] + 9[Cx^2 + Dx + E] = 27x^2$$

$$\underbrace{9C}_{27} x^2 + \underbrace{(9D - 12C)}_{0} x + \underbrace{(2C - 6D + 9E)}_{0} = 27x^2$$

3 krav:
1) $9C = 27 \Rightarrow C = 3$
2) $9D - 12C = 0 \Rightarrow D = \frac{12C}{9} = 4$
3) $2C - 6D + 9E = 0 \Rightarrow E = 2$

$$\Rightarrow y_p = 3x^2 + 4x + 2$$

Løsning av homogen ligning kjent fra a):

$$y_h = A \cdot e^{3x} + Bx \cdot e^{3x}$$

Generell løsning av inhomogen ligning:

$$y = y_h + y_p$$

$$y = A \cdot e^{3x} + Bx \cdot e^{3x} + 3x^2 + 4x + 2$$

Trenger den deriverte for å løse initialverdiproblemet:

$$y' = 3A \cdot e^{3x} + B \cdot e^{3x} + 3Bx e^{3x} + 6x + 4$$

Initialkravene:

$$y(0) = A + 2 = 4 \Rightarrow A = \underline{2}$$

$$y'(0) = 3A + B + 4 = 7$$

$$\Rightarrow B = 7 - 4 - 3A = \underline{-3}$$

Initialverdiproblemet har løsning:

$$y = \underline{\underline{2 \cdot e^{3x} - 3x \cdot e^{3x} + 3x^2 + 4x + 2}}$$

5 a)

i) Utvikler determinanten langs 3. rad:

$$|A| = \begin{vmatrix} 1 & 2 & 4 \\ 2 & 1 & -10 \\ 3 & 4 & 0 \end{vmatrix} = 3\begin{vmatrix} 2 & 4 \\ 1 & -10 \end{vmatrix} - 4\cdot\begin{vmatrix} 1 & 4 \\ 2 & -10 \end{vmatrix} + 0$$

$$= 3\left(2(-10) - 1\cdot 4\right) - 4\left(1\cdot(-10) - 4\cdot 2\right) = -72 + 72$$

$$= \underline{\underline{0}}$$

ii) A^{-1} eksisterer ikke siden $\det A = 0$

iii)

ok!

$(3\times 3)\ (3\times 2)$

$$\begin{bmatrix} 2 & 1 \\ -1 & 1 \\ \pi & 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 1 & -10 \\ 3 & 4 & 0 \end{bmatrix}\begin{bmatrix} 2-2+4\pi & 1+2+4 \\ 4-1-10\pi & 2+1-10 \\ 6-4 & 3+4+0 \end{bmatrix} = \underline{\underline{\begin{bmatrix} 4\pi & 7 \\ 3-10\pi & -7 \\ 2 & 7 \end{bmatrix}}}$$

ok!

$(2\times 3)\ (3\times 3)$

iv) $B^T A^T$: definert og er 2×3 matrise.

Raskest å bruke formelen $(AB)^T = B^T A^T$:

$$B^T A^T = (AB)^T$$

$$= \begin{bmatrix} 4\pi & 7 \\ 3-10\pi & -7 \\ 2 & 7 \end{bmatrix}^T = \underline{\underline{\begin{bmatrix} 4\pi & (3-10\pi) & 2 \\ 7 & -7 & 7 \end{bmatrix}}}$$

5b)

Merk at koeffisientmatrisen til lig. sys. er matrisen A fra oppg. a). Siden $|A| = 0$ må ligningsystemet ha enten ingen løsninger (inkonsistent) eller uendelig mange (fri variabel). Radreduserer totalmatrisen for å finne ut?

$$\begin{array}{cccc} x & y & z & h.s \end{array}$$
$$\begin{bmatrix} 1 & 2 & 4 & 6 \\ 2 & 1 & -10 & 8 \\ 3 & 4 & 0 & 6 \end{bmatrix} \underset{III - 3\cdot I}{\overset{II - 2\cdot I}{\sim}} \begin{bmatrix} 1 & 2 & 4 & 6 \\ 0 & -3 & -18 & -4 \\ 0 & -2 & -12 & -12 \end{bmatrix}$$

$$\overset{II \leftrightarrow III}{\sim} \begin{bmatrix} 1 & 2 & 4 & 6 \\ 0 & 2 & 12 & 12 \\ 0 & 3 & 18 & 4 \end{bmatrix} \overset{II\cdot\frac{1}{2}}{\sim} \begin{bmatrix} 1 & 2 & 4 & 6 \\ 0 & 1 & 6 & 6 \\ 0 & 3 & 18 & 4 \end{bmatrix}$$

$$\overset{III - 3\cdot II}{\sim} \begin{bmatrix} 1 & 2 & 4 & 6 \\ 0 & 1 & 6 & 6 \\ 0 & 0 & 0 & -14 \end{bmatrix} \overset{III\cdot\left(-\frac{1}{14}\right)}{\sim}$$
$$\begin{array}{cccc} x & y & z & h.s \end{array}$$
$$\begin{bmatrix} 1 & 2 & 4 & 6 \\ 0 & 1 & 6 & 6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

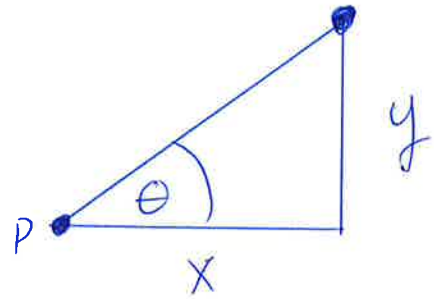
Siste rad tilsvarer ligningen

$$0\cdot x + 0\cdot y + 0\cdot z = 1$$

så ligningssystemet er inkonsistent.

Oppg. 6 Velger P i origo:

i)



$$\Rightarrow \tan\theta = \frac{y}{x}$$

$$\frac{d}{dt}\tan\theta = \frac{d}{dt}\frac{y}{x}$$

$$\frac{d\theta}{dt}\frac{d}{d\theta}\tan\theta = \frac{\frac{dy}{dt}\cdot x - y\cdot\frac{dx}{dt}}{x^2}$$

$$\theta'\cdot(1+\tan^2\theta) = \frac{y'\cdot x - y x'}{x^2}$$

ii) Setter inn:

$$\theta = 60^\circ, \quad x = 5\tfrac{m}{s}\cdot 20s = 100\ m$$

$$y = x\cdot\tan 60 = 100\sqrt{3}\ m$$

$$x' = 5\ m/s$$

$$\theta' = \frac{\pi}{180}\ rad/s$$

$$\Rightarrow \frac{\pi}{180}(1+\tan^2 60) = \frac{y'\cdot 100 - 100\sqrt{3}\cdot 5}{100^2}$$

$$\frac{\pi}{180}(1+3)\cdot 100 = y' - 5\sqrt{3}$$

$$\Rightarrow \quad y' = \frac{20}{9}\pi + 5\sqrt{3} \simeq 15{,}6$$

Vertikalhastigheten er ca. $\underline{\underline{15{,}6 \text{ m/s}}}$

i dette tidspunktet.